Limit Theorems for Sums and Maxima from Domains of Geometric Partial Attraction of Semistable and Max-Semistable Laws

Doktori értekezés

Megyesi Zoltán

Matematika és Számítástudományok Doktori Iskola
Szegedi Tudományegyetem, Bolyai Intézet
Sztochasztika Tanszék

Témavezető:
Dr. Csörgő Sándor

2002

1. Introduction

The notion of a 'semistable' distribution first appears in Lévy's classical work [57] as a natural generalization of the class $\mathcal{S}$ of stable laws. Consider the equation

$$c \log \phi(t) = \log \phi(qt) + ibt\,, \qquad t \in \mathbb{R}\,, \tag{1.1}$$

where ϕ is a proper characteristic function, $c > 1$, $q > 1$. If ϕ is a characteristic function belonging to a stable law, then for every $c > 1$ there exist $q > 1$ and $b \in \mathbb{R}$ such that (1.1) holds. If (1.1) does not necessarily hold for every $c > 1$ but there exists *some* $c > 1$ such that (1.1) holds for some $q > 1$ and b pertaining to this special c, the class of characteristic functions satisfying this condition is the class of semistable characteristic functions: the distributions determined by the elements of this class are the semistable laws. The set of semistable laws is denoted by $\mathcal{S}_*$. Lévy's original definition stipulated $b = 0$, but in almost all cases this is not a real restriction. Starting from definition, Lévy [57] described the canonical (Lévy-Khintchin) representation of semistable characteristic functions. The following example is also due to him:

Example 1. *(P. Lévy)* The characteristic function given by

$$\log \phi(z) := \sum_{\nu=-\infty}^{\infty} \frac{\cos zq^\nu - 1}{q^\nu}$$

is semistable.

Indeed, $\phi(z)$ 'has the curious property' that $\phi^q(z) = \phi(qz)$, i.e., $\phi(z)$ satisfies (1.1) with $c = q$ and $b = 0$. The original source is [57], p. 208, and it is quoted by Feller [40], p. 538, in the special case of $q = 2$.

Since stable distributions arise as limiting distributions of sums of suitably centred and normalized independent identically distributed random variables along the whole $\mathbb{N} = \{1, 2, \dots\}$, it is intuitively not surprising that semistable distributions may arise as limit distribution of independent identically distributed random variables along subsequences satisfying some restrictions (which will be our definition). Indeed, Doeblin in 1940 already remarked in a footnote ([38], p. 79), without any proof or further discussion that semistable distributions arise if the *normalizing constants* satisfy a geometric growth condition, the precise formulation of the results took place only 30 years later. Between 1970 and 1973 four authors, independently of one other, arrived to similar results: however, the precise formulation of the conditions and results requires some notation, which will be used throughout.

Let $X_1, X_2, \dots$ be a sequence of independent, identically distributed random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with a common distribution function $F(x) :=$

$\mathbb{P}\{X_1 \le x\}$, $x \in \mathbb{R}$, and for each positive integer $n \in \mathbb{N}$ introduce $X_{1,n} \le \ldots \le X_{n,n}$, the order statistics based on $X_1, \ldots, X_n$. We say that F is in the domain of partial attraction of the infinitely divisible distribution function G, written $F \in \mathbb{D}_{\mathrm{p}}(G)$ if for the random variable V with distribution function G there exists a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ and centring and normalizing constants $B_{k_n} \in \mathbb{R}$ and $A_{k_n} > 0$ such that

$$\frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{k_n} X_j - B_{k_n} \right\} \xrightarrow{\mathcal{D}} V, \tag{1.2}$$

where and in the sequel a convergence relation is meant to hold as $n \to \infty$ unless otherwise specified. For a sequence $\{V_n\}$ of random variables and a random variable V we write $V_n \xrightarrow{\mathcal{D}} V$ to denote convergence in distribution, i.e., the convergence of the distribution functions of V_n to the distribution function of V at each continuity point of the latter, and $\xrightarrow{\mathbb{P}}$ to denote convergence in probability; also, $\stackrel{\mathcal{D}}{=}$ will denote equality in distribution. Also, if $V_n \xrightarrow{\mathcal{D}} V$ and V has distribution function G, then we will also write $V_n \xrightarrow{\mathcal{D}} G$, hoping that this little confusion in the notation will not lead to any misunderstanding. Consider now the following (semi-) geometric conditions on the growth rate of the subsequence $\{k_n\}$:

$$\liminf_{n\to\infty} k_{n+1}/k_n = c, \qquad 1 < c < \infty \; ; \tag{1.3a}$$

$$\limsup_{n\to\infty} k_{n+1}/k_n = c, \qquad 1 \le c < \infty \; ; \tag{1.3b}$$

$$\lim_{n\to\infty} k_{n+1}/k_n = c, \qquad 1 \le c < \infty \, . \tag{1.3c}$$

Observe that (1.3c) is trivially satisfied for the geometric subsequence $k_n = \lfloor c^n \rfloor$, $c > 1$, where and in the sequel $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the lower and the upper integer part, respectively. Obviously a subsequence $\{k_n\}$ satisfying (1.3c) satisfies (1.3b), furthermore, a subsequence $\{k_n\}$ satisfying (1.3b) contains a further subsequence $\{k_n'\} \subset \{k_n\}$ such that (1.3a) holds along $\{k_n'\}$ with some constant $c' > 1$. (Note that the condition $c > 1$ is essential in (1.3a).) Since if (1.2) holds along $\{k_n\}$ then it holds along $\{k_n'\} \subset \{k_n\}$ just as well, (1.3c) appears to be the strongest and (1.3a) the weakest restriction. Now, we are in the position to give the definition of a semistable law without any reference to characteristic functions.

Definition. *Non-degenerate distributions that arise as limiting distributions of suitably centred and normalized sums of independent identically distributed random variables along subsequences satisfying (1.3c) are called semistable laws.*

In fact, as it will be shown by Theorem 2.1, conditions (1.3a)–(1.3c) turn out to be equivalent with respect to the determination of the limiting laws. Of course, stable laws are semistable and all semistable laws are infinitely divisible.

Returning to history, the first author who links the growth rate of the subsequence to the defining property of the characteristic function in (1.1) is Shimizu. Shimizu [70] considers the sequence $\lfloor c^n \rfloor$, $c > 1$, i.e., the geometric subsequence and shows among others that the arising limiting distributions along such a subsequence are semistable. Shimizu approaches the problem from the point of view of functional equations on characteristic functions. Pillai's short note [67] deals with the case of the geometric subsequence again. Both Pillai and Shimizu refer to Lévy.

Kruglov [55] requires the growth condition in (1.3c) while Mejzler [63] allows more general conditions like (1.3a)–(1.3b). Both authors obtain the Lévy characterization of a semistable law and a statement similar to our Corollary 2.4 below on the moments of the limiting and the attracted distributions. Kruglov also proves a local limit theorem, while Mejzler carries out a detailed investigation of domains of partial attraction. Neither Kruglov nor Mejzler refers to Lévy nor uses the semistable terminology — Mejzler refers to Feller, who ([40], p. 538) quotes the aforementioned example of Lévy, but not to Lévy itself. The next example of a semistable characteristic function is due to Mejzler:

Example 2. *(D. Mejzler)* The characteristic function given by

$$\log \phi(t) = -|t|^{1/4} \exp\left(\frac{\sin \log |t|}{64}\right)$$

satisfies (1.1) with $c = \exp(\pi/2)$, $q = c^4$ and $b = 0$.

Multidimensional generalizations of semistability started to appear in the late 1970's. For recent results in this direction see Meerschaert and Scheffler [60] and the references contained therein.

The characterization of the attracted distributions took place in the mid-nineties, in Grinevich and Khoklov [49]. However, since this characterization contained a small error, we believe that our one, happening in terms of quantile functions in Theorem 3 in [61], here stated as Theorem 2.3, is the first mathematically correct description of these distributions. The results are formulated in terms of distribution functions in Corollary 2.3.

The simplest example for a distribution attracted in geometrical order (i.e., along a subsequence $\{k_n\}$ satisfying (1.3c)) to a semistable law is provided by the 289-year-old St. Petersburg game. This example also shows that semistable laws arise not only as results of sophisticated constructions but in a very natural way.

Example 3. *(The St. Petersburg game)* Peter flips a coin until the first 'head' and Paul wins 2^k *ducats* if this happens at the k-th flip. Denoting Paul's gain by X, for the distribution function F of X we have

$$F(x) = P\{X \leq x\} = \begin{cases} 0, & x < 2, \\ 1 - 2^{-\lfloor \mathrm{Log} x \rfloor}, & x \geq 2, \end{cases} \tag{1.4}$$

where Log stands for the logarithm to the base 2 So far the game is very favourable for Paul, and it is quite natural that Peter asks Paul to pay an entrance fee for this game. What is the 'fair price' Paul has to pay for this game? We immediately see that $E(X) = \infty$, therefore the usual consideration, suggesting that the 'fair price' should equal $E(X)$, fails. This is the well-known St. Petersburg paradox, first mentioned in 1713. Instead of a single game, naturally one has to consider Paul's cumulative gain $S_n := X_1 + X_2 + \cdots + X_n$. However, S_n fails to converge in distribution along the entire $\mathbb{N}$ with any centralization and normalization: instead, as it was proved by Martin-Löf [58], there exist centralization and normalization such that S_{k_n} converges along $k_n = 2^n$, a subsequence obviously satisfying (1.3c). We see therefore that the St. Petersburg distribution function is attracted to some semistable law. For an enjoyable account of both the history and the resolution of the St. Petersburg paradox see Csörgő [21]. The distribution function in (1.4) turns out to be interesting also in a different aspect — see the Counterexample in Section 4.2.

Not only the convergence of sums but that of maxima is discussed in the literature. Introducing $M_n := X_{n,n} = \max\{X_1, \ldots, X_n\}$, the possible limiting distributions of the sequence $b_n^{-1}(M_n - a_n)$ were determined by Gnedenko [43] in the early forties (see also the monographs de Haan [50], Galambos [42] and Resnick [68]) — these distributions are called the extreme value distributions, or, in analogy with the case of sums, max-stable. If the normalized and centered maxima of the independent identically distributed random variables converge along subsequences that satisfy (1.3c), the arising limiting laws are exactly the max-semistable laws. More specifically, the conditions are that

$$\mathbb{P}\left(b_{k_n}^{-1}(M_{k_n} - a_{k_n}) \le x\right) = \Big(F\big(b_{k_n}x + a_{k_n}\big)\Big)^{k_n} \to G(x),$$

for all $x \in C_G$. Here C_G denotes the set of continuity points of the distribution function G and the eventually strictly increasing subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ is assumed to satisfy (1.3c). Max-semistable laws were first introduced in 1992 by Grinevich [46] and Pancheva [65]. One of the natural examples of a distribution function attracted to a max-semistable law (in the sense of maxima, of course) is the St. Petersburg distribution function again. Another simple example is the geometric distribution.

Example 4. *(The geometric distribution)* Let $X_1, X_2, \ldots$ be distributed according to the geometric distribution with parameter p, i. e., $\mathbb{P}(X_1 = k) = pq^{k-1}$, $k \in \mathbb{N}$, where $q = 1 - p$. Then

$$F(x) = 1 - q^{\lfloor x \rfloor}, \qquad x \ge 0.$$

It can be checked directly that $F^n(b_n x + a_n)$ does not converge in distribution to any non-degenerate limiting law for any normalization $\{b_n\}$ and centralization $\{a_n\}$. However, setting $k_n := \lfloor q^{-n} \rfloor$, $b_{k_n} :\equiv \frac{-1}{\log q}$ and $a_{k_n} := n$,

$$\Big(F\big(b_{k_n}x + a_{k_n}\big)\Big)^{k_n} = \left(1 - \frac{q^{\lfloor \frac{-x}{\log q} \rfloor}}{q^{-n}}\right)^{\lfloor q^{-n} \rfloor} \to \exp\Big(-q^{\lfloor \frac{-x}{\log q} \rfloor}\Big), \tag{1.5}$$

at any continuity point of the limiting distribution. Since condition (1.3c) is evidently satisfied for this $\{k_n\}$, we see that the limiting distribution here is max-semistable and the geometric distribution is attracted to it along a geometrically growing subsequence.

In what follows we intend to give a systematic treatment of the theory of semistable and max-semistable laws. Besides investigating the (subsequential) convergence of the sums $S_{k_n}(0,0) := \sum_{j=1}^{k_n} X_j$, we will also deal with the convergence of the lightly trimmed sums $S_{k_n}(l,m) := \sum_{j=l+1}^{k_n-m} X_{j,k_n}$, where l and m are some fixed nonnegative integers, and of the moderately trimmed ones $S_n(l_n, m_n) = \sum_{j=l_n+1}^{n-m_n} X_{j,n}$, where $l_n \to \infty$, $\frac{l_n}{n} \to 0$ and $m_n \to \infty$, $\frac{m_n}{n} \to 0$. Both lightly trimmed and moderately trimmed sums are basic objects in the statistical theory of robust estimation. At this aim, we will work within the framework of the "probabilistic approach" of Csörgő, Haeusler, and Mason [26]. This approach is based upon the asymptotic behaviour of the uniform empirical distribution function in conjunction with the tail properties of the underlying quantile function. The theory thus obtained was completed in [19] by a study of domains of partial attraction; further details are in [22–25] and [35]. Since untrimmed (whole) sums can be regarded as the special case $l = m = 0$ of lightly trimmed ones, it is conceivable that the behaviour of the two are closely connected, even though lightly trimmed sums are in general very hardly accessed by classical Fourier analytic tools. An immediate advantage of the 'probabilistic' approach is that it is capable to deal with both cases in a unified frame. For easier reference, the rudiments of this approach will be summarized at the beginning of Section 2.

To deal with moderately trimmed sums, we will use in Section 3 the tools supplied by Csörgő, Haeusler, and Mason [27]: the necessary background is collected again at the beginning of that section. Section 4 is devoted to the study of max-semistable laws and their domains of geometric partial attraction. The results on semistable laws in Section 2 and on max-semistable laws in Section 4 are applied to obtain almost sure limit theorems in Section 5.

2. Full and lightly trimmed sums

2.1. Preliminaries

Let Ψ be the class of all non-positive, non-decreasing, right-continuous functions $\psi(\cdot)$ defined on the positive half-line $(0,\infty)$ such that

$$\int_{\varepsilon}^{\infty} \psi^2(s)\,ds < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{2.1}$$

Concerning the integral sign the following convention will be used: When $0 \le a < b \le \infty$, l is a left-continuous and r is a right-continuous function then $\int_a^b r\,dl = \int_{[a,b)} r\,dl$ and $\int_a^b l\,dr = \int_{(a,b]} l\,dr$, whenever these integrals make sense as Lebesgue–Stieltjes integrals. In this case the usual integration by parts formula holds. Let $E_1^{(j)}, E_2^{(j)}, \ldots; j = 1,2$, be two independent sequences of independent exponentially distributed random variables with mean 1 and with their partial sums $Y_n^{(j)} = E_1^{(j)} + \cdots + E_n^{(j)}$, $n \ge 1$, $j = 1,2$, as jump points, consider the standard left-continuous independent Poisson processes $N_j(u) := \sum_{n=1}^{\infty} I\big(Y_n^{(j)} < u\big)$, $0 \le u < \infty$, $j = 1,2$, where $I(\cdot)$ is the indicator function. For a function $\psi \in \Psi$, consider the random variables

$$W_j(\psi) := \int_1^{\infty} (N_j(s) - s)\,d\psi(s) + \int_0^1 N_j(s)\,d\psi(s) + \Theta(\psi), \quad j = 1,2,$$

and

$$\begin{aligned} W_j^{(l)}(\psi) := &\int_{Y_{l+1}^{(j)}}^{\infty} (N_j(s) - s)\,d\psi(s) - \int_1^{Y_{l+1}^{(j)}} s\,d\psi(s) + l\psi(Y_{l+1}^{(j)}) \\ &- \int_1^{l+1} \psi(s)\,ds - \psi(1), \quad j = 1,2, \end{aligned}$$

where the first integrals are almost surely well-defined by condition (2.1) as improper Riemann integrals and where $l \ge 0$ is an arbitrary integer and

$$\Theta(\psi) := -\psi(1) + \int_0^1 \frac{\psi(s)}{1+\psi^2(s)}\,ds - \int_1^{\infty} \frac{\psi^3(s)}{1+\psi^2(s)}\,ds.$$

Let Z be a standard normal random variable such that $N_1(\cdot)$, Z, and $N_2(\cdot)$ are independent, let $\sigma \ge 0$ be a finite constant and for $\psi_1, \psi_2 \in \Psi$ introduce the variables

$$V_{l,m}(\psi_1,\psi_2,\sigma) := -W_1^{(l)}(\psi_1) + \sigma Z + W_2^{(m)}(\psi_2)$$

and

$$V(\psi_1,\psi_2,\sigma) := -W_1(\psi_1) + \sigma Z + W_2(\psi_2), \tag{2.2}$$

where l and m are non-negative integers, so that

$$V(\psi_1,\psi_2,\sigma) = V_{0,0}(\psi_1,\psi_2,\sigma) - \Theta(\psi_1) - \psi_1(1) + \Theta(\psi_2) + \psi_2(1).$$

Hence by Theorem 3 in [26], for all $t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}\Big(e^{itV(\psi_1,\psi_2,\sigma)}\Big) = \exp\bigg\{ -\frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0} \Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big) dL(x) \\ + \int_0^\infty \Big(e^{itx} - 1 - \frac{itx}{1+x^2}\Big) dR(x) \bigg\}, \end{aligned} \tag{2.3}$$

where $L(x) := \inf\{s > 0 : \psi_1(s) \geq x\}$, $x < 0$, and $R(x) := -\inf\{s > 0 : \psi_2(s) \geq -x\}$, $x > 0$, $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and $R(\cdot)$ is right-continuous and non-decreasing on $(0,\infty)$ with $R(\infty) = 0$ and by (2.1), $\int_{-\varepsilon}^{0} x^2 dL(x) + \int_0^\varepsilon x^2 dR(x) < \infty$ for every $\varepsilon > 0$. Here $V(\psi_1,\psi_2,\sigma)$, or, more generally by Theorem 4 in [26], $V_{l,m}(\psi_1,\psi_2,\sigma)$ is degenerate if and only if $(\psi_1,\psi_2,\sigma) = (0,0,0)$. Thus, modulo an additive constant, $V(\psi_1,\psi_2,\sigma)$ represents an arbitrary infinitely divisible random variable by the well-known Lévy characterization of the characteristic function of infinitely divisible laws (p. 84 in [44]), and this representation is unique (and serves as a basis of approximations of infinitely divisible distributions themselves in [20] as part of the 'probabilistic' theory). Therefore, modulo an additive constant, the set $\mathcal{I} := \{(\psi_1,\psi_2,\sigma) \neq (0,0,0) : \psi_1,\psi_2 \in \Psi, \sigma \geq 0\}$ can be identified with the class of all non-degenerate infinitely divisible distributions.

Define the left-continuous inverse or quantile function pertaining to F by $Q(u) := \inf\{x : F(x) \geq u\}$, $0 < u \leq 1$, and its right-continuous version by $Q_+(u) := \lim_{s\downarrow u} Q(s)$, $0 \leq u < 1$, furthermore, set $Q(0) := Q_+(0)$ and $Q_+(1) := Q(1)$. For $0 < s < 1-t < 1$ consider the function

$$\begin{aligned} \sigma^2(s, 1-t) &= \int_s^{1-t}\int_s^{1-t} \big[\min(u,v) - uv\big]\, dQ(u)\, dQ(v) \\ &= sQ^2(s) + tQ^2(1-t) + \int_s^{1-t} Q^2(u)\, du \\ &\quad - \bigg[sQ(s) + tQ(1-t) + \int_s^{1-t} Q(u)\, du\bigg]^2, \end{aligned} \tag{2.4}$$

basic in [26], and for $0 < s < 1/2$ simply put $\sigma(s) := \sigma(s, 1-s)$. Then for all $n \in \mathbb{N}$ large enough, the quantity

$$a(n) := n^{1/2}\sigma(1/n) \tag{2.5}$$

is positive. In the sequel $a(n)$ will be referred to as the 'natural' normalizing sequence. Introduce also the 'natural' centring sequence for any fixed pair of nonnegative integers l, m:

$$\mu_{l,m}(n) := \int_{(l+1)/n}^{1-(m+1)/n} Q_+(u)\, du. \tag{2.6}$$

We shall need the right-continuous functions

$$\psi_1(n;s) := \begin{cases} Q_+\left(\frac{s}{n}\right)/a(n), & 0 < s \le n - n\alpha_n\,, \\ Q_+(1-\alpha_n)/a(n), & n - n\alpha_n < s < \infty\,, \end{cases}$$
$$\psi_2(n;s) := \begin{cases} -Q\left(1-\frac{s}{n}\right)/a(n), & 0 < s \le n - n\alpha_n\,, \\ -Q(\alpha_n)/a(n), & n - n\alpha_n < s < \infty\,, \end{cases} \tag{2.7}$$

where $n \in \mathbb{N}$ and $\alpha_n > 0$ is any sequence such that $\alpha_n \downarrow 0$ and $n\alpha_n \to 0$ as $n \to \infty$. According to the main results of [26] and [19], the convergence of the lightly trimmed or untrimmed sums $\sum_{j=l}^{n-m} X_{j,n}$, possibly along subsequences of $\mathbb{N}$, is governed by the functions $\psi_j(n;\cdot)$, $j = 1,2$, along the same subsequences, and by the behaviour of the function $\sigma(\cdot\,,\cdot)$. It also follows from Lemma 2.5 in [26] that if there is a subsequence $\{n'\} \subset \mathbb{N}$, a sequence of numbers $A(n') > 0$ along this subsequence, and right-continuous functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ on $(0,\infty)$ which are necessarily non-positive and non-decreasing such that $a(n')\psi_j(n';\cdot)/A(n') \Longrightarrow \psi_j(\cdot)$ as $n' \to \infty$, $j = 1,2$, and the sequence $\{a(n')/A(n')\}$ is bounded, then both $\psi_1(\cdot)$ and $\psi_2(\cdot)$ satisfy (2.1), and hence $\psi_1, \psi_2 \in \Psi$. Here and in the sequel $\Longrightarrow$ denotes weak convergence on the positive half-line, i.e., convergence at each continuity point of the limiting function and all subsequences of $\mathbb{N}$ will be assumed to be *unbounded* and those denoted by $\{k_n\}$ are also assumed to be *eventually strictly increasing* (there is an n_0 such that $k_{n+1} > k_n$ for $n > n_0$).

Owing to the identification of the set of triplets in $\mathcal{I}$ and the class of nondegenerate infinitely divisible distributions, the random variable on the right-hand side of (1.2) is necessarily of the form $V = V(\psi_1,\psi_2,\sigma)$ and its distribution function $G = G_{\psi_1,\psi_2,\sigma}$, $(\psi_1,\psi_2,\sigma) \in \mathcal{I}$. Instead of $F \in \mathbb{D}_{\mathrm{p}}(G_{\psi_1,\psi_2,\sigma})$ we will often simply write $F \in \mathbb{D}_{\mathrm{p}}(\psi_1,\psi_2,\sigma)$ to denote that F is in the domain of partial attraction of the infinitely divisible distribution function $G_{\psi_1,\psi_2,\sigma}$, $(\psi_1,\psi_2,\sigma) \in \mathcal{I}$, and (1.2) can be rewritten as

$$\frac{1}{A_{k_n}}\left\{\sum_{j=1}^{k_n} X_j - B_{k_n}\right\} \xrightarrow{\mathcal{D}} V(\psi_1,\psi_2,\sigma). \tag{2.8}$$

A well-known theorem of Khintchin states that $\mathbb{D}_{\mathrm{p}}(\psi_1,\psi_2,\sigma) \neq \emptyset$ for any $(\psi_1,\psi_2,\sigma) \in \mathcal{I}$, and all possible nondegenerate limiting distributions in (2.8) are necessarily infinitely divisible.

If we stipulate $\{k_n\} = \{n\} = \mathbb{N}$ in (2.8) then by classical theory the arising limiting distribution is stable (see Gnedenko and Kolmogorov [44] or Corollary 5* in [19]), and we have either $(\psi_1,\psi_2,\sigma) = (0,0,\sigma)$ for some $\sigma > 0$, in which case we say that F is in the domain of attraction of the normal distribution, written $F \in \mathbb{D}(2)$, or $(\psi_1,\psi_2,\sigma) =$

$(m_1\psi^\alpha, m_2\psi^\alpha, 0)$ for some constants $\alpha \in (0,2)$, $m_1, m_2 \geq 0$, $m_1 + m_2 > 0$, where $\psi^\alpha(s) = -s^{-1/\alpha}$, $s > 0$, in which case F is in the domain of attraction of a stable distribution of exponent α, written $F \in \mathbb{D}(\alpha)$; furthermore, every stable distribution arises this way. As it is suggested by the notation just introduced, the Gaussian law is regarded as stable law with exponent 2. (The superscript α in ψ^α, and in ψ_1^α and ψ_2^α beginning with Theorem 2.1 below, is meant as a label, not as a power exponent.)

2.2. The characterization of semistable laws

Theorem 2.1. (Characterization theorem) *Suppose that (2.8) holds along some subsequence $\{k_n\}$ satisfying (1.3a). Then, necessarily, either*

$$\sigma > 0 \qquad \textit{and} \qquad \psi_1 = \psi_2 \equiv 0, \tag{2.9}$$

in which case the limiting distribution in (2.8) is normal, or

$$\begin{aligned} &\sigma = 0 \qquad \textit{and} \\ &\psi_1(s) = \psi_1^\alpha(s) := -s^{-1/\alpha} M_1(s), \\ &\psi_2(s) = \psi_2^\alpha(s) := -s^{-1/\alpha} M_2(s), \end{aligned} \tag{2.10}$$

$0 < s < \infty$, where $\alpha \in (0,2)$, M_1 and M_2 are non-negative, right-continuous functions, at least one of which is not identically 0. Furthermore, if $M_j \not\equiv 0$ then M_j is bounded away from both 0 and ∞, and, for the $c > 1$ figuring in (1.3a), satisfies

$$M_j(cs) = M_j(s), \qquad 0 < s < \infty, \tag{2.11}$$

and for $0 < s_0 < s_1 < \infty$,

$$s_0^{-1/\alpha} M_j(s_0) \geq s_1^{-1/\alpha} M_j(s_1), \qquad j = 1, 2. \tag{2.12}$$

Conversely, suppose that for the infinitely divisible random variable $V(\psi_1, \psi_2, \sigma)$ we have $\sigma > 0$ and $\psi_1 \equiv \psi_2 \equiv 0$, or $\sigma = 0$, $\psi_1 = \psi_1^\alpha$ and $\psi_2 = \psi_2^\alpha$ with ψ_1^α and ψ_2^α defined above, at least one of which is not identically 0, and for the c figuring in (2.11) we have $c > 1$. Then, in case of $\sigma = 0$ there exists a distribution function F such that (2.8) holds along a subsequence $\{k_n\}$ satisfying (1.3c) with the given c, while in case of $\sigma > 0$ the distributional convergence in (2.8) is satisfied for the underlying F even along the whole $\mathbb{N}$.

The property in (2.11) will be referred to as multiplicative periodicity with period c. A comparison of the necessity and sufficiency parts of the above theorem shows that conditions (1.3a)–(1.3c) are equivalent with respect to the determination of the arising limiting distributions. Hence next definition of a semistable law is identical in content with the one given in Section 1.

Definition*. *Infinitely divisible distributions which arise as limiting distributions of suitably centred and normalized sums of independent identically distributed random variables along subsequences satisfying one of the conditions in (1.3a)–(1.3c) are called semistable laws.*

Clearly, the conditions on the functions M_1 and M_2 in Theorem 2.1 and $\alpha \in (0,2)$ imply that $\psi_1^\alpha, \psi_2^\alpha \in \Psi$. Comparing (2.10) and the representation of a stable law in the preceding subsection, one can see that every stable law is semistable corresponding to the case $M_j \equiv m_j$, $j = 1,2$, $m_1, m_2 \geq 0$, $m_1 + m_2 > 0$. If $\{k_n\} = \mathbb{N}$, then for any given $c > 1$ we can choose a subsequence such that (1.3a) holds along it with this given c; thus we obtain that $M_j(cs) = M_j(s)$, $0 < s < \infty$, for any $c > 1$, i.e., M_j is necessarily constant, $j = 1,2$. In a similar manner as in the stable case, it makes sense to talk about the constant α appearing in (2.10) as the *exponent* of a semistable law; we set $\alpha = 2$ if $\sigma > 0$.

Since for the infinitely divisible random variable $V(\psi_1, \psi_2, \sigma)$ the corresponding Lévy measures in (2.3) can be determined by an inversion, from Theorem 2.1 it is not difficult to deduce the Lévy measures pertaining to semistable laws. Clearly, it is enough to deal with the non-normal case, i.e., when $\alpha \in (0,2)$ and the random variable is of the form $V(\psi_1^\alpha, \psi_2^\alpha, 0)$. We obtain that a non-normal infinitely divisible distribution is semistable if and only if for its Lévy functions L and R there exist two non-negative bounded functions $M_L(\cdot)$ on $(-\infty, 0)$ and $M_R(\cdot)$ on $(0,\infty)$, one of which has a strictly positive infimum and the other one either has a strictly positive infimum or is identically zero, such that $M_L(c^{1/\alpha} x) = M_L(x)$, $x < 0$, and $M_R(c^{1/\alpha} x) = M_R(x)$, $x > 0$, where c is the same as in Theorem 2.1, and the functions $L(x) = M_L(x)/|x|^\alpha$, $x < 0$ and $R(x) = -M_R(x)/x^\alpha$, $x > 0$, are left-continuous and non-decreasing on $(-\infty, 0)$ and right-continuous and non-decreasing on $(0, \infty)$, respectively.

An immediate advantage of our 'probabilistic' approach is that we are able to determine how the central and extremal elements of the sample contribute to the arising limiting distribution. The assertion below follows from a combination of Theorem 2.1 above and Theorems 1* and 6 in [19].

Corollary 2.1. *Assume that (2.8) holds along some subsequence $\{k_n\}$ satisfying (1.3a) with some random variable $V(\psi_1, \psi_2, \sigma)$ necessarily semistable. Then there exists a subsequence $\{r_n\} \subset \mathbb{N}$ such that $r_n/k_n \to 0$, and,*

(i) *if $V(\psi_1, \psi_2, \sigma)$ is normal, i.e., $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$, then for any fixed pair of nonnegative integers l and m*

$$\frac{1}{a(k_n)} \left\{ \sum_{j=l+1}^{k_n - m} X_{j,k_n} - k_n \mu_{l,m}(k_n) \right\} \xrightarrow{\mathcal{D}} N(0,1),$$

where $N(\mu, \sigma^2)$ denotes a Gaussian random variable with mean μ and variance σ^2, and

$a(k_n)$ and $\mu_{l,m}(k_n)$ are as in (2.5) and (2.6). In this case even

$$\frac{1}{a(k_n)}\left\{\sum_{j=r_n+1}^{k_n-r_n} X_{j,k_n} - k_n\int_{(r_n+1)/k_n}^{1-(r_n+1)/k_n} Q_+(u)du\right\} \xrightarrow{\mathcal{D}} N(0,1),$$

i.e., the limiting distribution arises from the central elements of the sample.

(ii) *If $V(\psi_1,\psi_2,\sigma)$ is non-normal, i.e., $(\psi_1,\psi_2,\sigma)=(\psi_1^\alpha,\psi_2^\alpha,0)$ then*

$$\frac{1}{a(k_n)}\left\{\sum_{j=r_n+1}^{k_n-r_n} X_{j,k_n} - k_n\int_{(r_n+1)/k_n}^{1-(r_n+1)/k_n} Q_+(u)du\right\} \xrightarrow{\mathbb{P}} 0$$

and for any subsequence $\{k_{n'}\}\subset\{k_n\}$ there exists $\{k_{n''}\}\subset\{k_{n'}\}$ such that

$$A_{k_{n''}}/a(k_{n''}) \to \beta \quad \text{as} \quad n''\to\infty$$

for some $\beta\in(0,\infty)$, and for any fixed pair of nonnegative integers l and m we have

$$\frac{1}{a(k_{n''})}\left\{\sum_{j=l+1}^{r_{n''}} X_{j,k_{n''}} - k_{n''}\int_{(l+1)/k_{n''}}^{(r_{n''}+1)/k_{n''}} Q_+(u)du\right\} \xrightarrow{\mathcal{D}} \beta W_1^{(l)}(\psi_1^\alpha),$$

$$\frac{1}{a(k_{n''})}\left\{\sum_{j=k_{n''}-r_{n''}+1}^{k_{n''}-m} X_{j,k_{n''}} - k_{n''}\int_{1-(r_{n''}+1)/k_{n''}}^{1-(m+1)/k_{n''}} Q_+(u)du\right\} \xrightarrow{\mathcal{D}} \beta W_2^{(m)}(\psi_2^\alpha),$$

as $n''\to\infty$. In this case the limiting distribution arises from the extremal elements and the middle portion of the sample vanishes. ∎

The statements of the corollary involve moderate trimming: Section 3 contains far-reaching generalizations and extensions of these results. In fact, part (ii) is just a special case of Theorem 3.3. However, this is not surprizing, since the tools we use in this section are optimized to deal with light trimming. Now we turn to the proofs.

Proofs

First we present three lemmas:

Lemma 2.1. *Assume that (2.8) holds for a limiting nondegenerate random variable V and some subsequence $\{k_n\}$. Then, for any sequence $\{c_n\}$ such that $c_n\to 0$,*

$$\frac{1}{A_{k_n}}\left\{\sum_{j=1}^{\lfloor c_n k_n\rfloor} X_j - c_n B_{k_n}\right\} \xrightarrow{\mathbb{P}} 0, \tag{2.13}$$

where the sum $\sum_{j=1}^{\lfloor c_n k_n \rfloor} X_j$ *is understood as* $-\sum_{j=1}^{\lceil |c_n k_n| \rceil} X_j$ *if* c_n *is negative.*

Proof. The proof does not necessarily need Fourier analytic tools; it can be proved using 'probabilistic' methods only. However, in this particular case the Fourier analytic proof is simpler.

Therefore, let $\phi(\cdot)$ and $\zeta(\cdot)$ denote the characteristic functions of the independent identically distributed random variables X_j and of the infinitely divisible random variable V, respectively. According to classical theory, (2.8) is equivalent to

$$\lim_{n\to\infty} \phi^{k_n}(t/A_{k_n}) \exp(-itB_{k_n}/A_{k_n}) = \zeta(t), \qquad t \in \mathbb{R}. \tag{2.14}$$

Introduce $\{n_1\} := \{n \in \mathbb{N} : c_n > 0\}$ and $\{n_2\} := \{n \in \mathbb{N} : c_n < 0\}$. Clearly, it suffices to consider the case when at least one of $\{n_1\}$, $\{n_2\}$ is unbounded. If $\{n_1\}$ is unbounded, then for the characteristic function of the left side in (2.13)

$$\begin{aligned}
&\phi^{\lfloor c_{n_1} k_{n_1} \rfloor}(t/A_{k_{n_1}}) \exp(-itc_{n_1} B_{k_{n_1}}/A_{k_{n_1}}) \\
&\qquad = \left[\phi^{k_{n_1}}(t/A_{k_{n_1}}) \exp(-itB_{k_{n_1}}/A_{k_{n_1}})\right]^{c_{n_1}} \phi^{\lfloor c_{n_1} k_{n_1} \rfloor / c_{n_1} k_{n_1}}(t/A_{k_{n_1}}) \to 1,
\end{aligned}$$

as $n_1 \to \infty$, for each $t \in \mathbb{R}$. Here the first factor on the right-hand side of the above equality converges to 1 by (2.14), since ζ is nowhere 0, and the second one does the same, since $A_{k_{n_1}} \to \infty$, $\phi(0) = 1$ and $\limsup_{n_1\to\infty} \lfloor c_{n_1} k_{n_1} \rfloor / c_{n_1} k_{n_1} \le 1$. Thus, because the constant 1 is the characteristic function of the degenerate random variable 0, we have shown that (2.13) holds along $\{n_1\}$. For $\{n_2\}$ a similar consideration applies. ∎

We shall need the following variant of Lemma 2.4 in [19]:

Lemma 2.2. *Suppose (2.8). Then for any pair of integers* $l, r \ge 1$,

$$\frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{\lfloor l k_n / r \rfloor} X_j - \frac{l}{r} B_{k_n} \right\} \xrightarrow{\mathcal{D}} V\left({}^{(l/r)}\psi_1, {}^{(l/r)}\psi_2, \sqrt{\frac{l}{r}}\,\sigma \right),$$

where for $\lambda > 0$ *and* $\psi \in \Psi$ *the function* ${}^{(\lambda)}\psi$ *is defined by*

$${}^{(\lambda)}\psi(s) := \psi\left(\frac{s}{\lambda}\right), \qquad s > 0.$$

Lemma 2.3. *Suppose (2.8). Then for any sequence* $c_n \to c > 0$ *we have*

$$\frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{\lfloor c_n k_n \rfloor} X_j - c_n B_{k_n} \right\} \xrightarrow{\mathcal{D}} V({}^{(c)}\psi_1, {}^{(c)}\psi_2, \sqrt{c}\,\sigma). \tag{2.15}$$

∎

Proof. Consider any sequence $\{\tau_m\}_{m=0}^{\infty}$ of rational numbers such that $\tau_m \uparrow c$. Then by Lemma 2.2, for any fixed m

$$\frac{1}{A_{k_n}}\left\{\sum_{j=1}^{\lfloor \tau_m k_n \rfloor} X_j - \tau_m B_{k_n}\right\} \xrightarrow{\mathcal{D}} V({}^{(\tau_m)}\psi_1, {}^{(\tau_m)}\psi_2, \sqrt{\tau_m}\,\sigma). \tag{2.16}$$

Let us denote the distribution functions of the random variables on the left-hand sides of (2.15) and (2.16) by F_{c_n} and F_{τ_m,k_n}, respectively. Accordingly, the distribution functions of the right-hand sides of (2.15) and (2.16) are denoted by G_c and G_{τ_m}.

For two distribution functions G and H introduce

$$\mathcal{L}(G,H) = \inf\{\varepsilon > 0 : G(x-\varepsilon) - \varepsilon \le H(x) \le G(x+\varepsilon) + \varepsilon \quad \text{for all} \quad x \in \mathbb{R}\},$$

their Lévy distance. It is well-known that this distance metrizes the weak convergence of distribution functions on the line (cf. p. 33 in [44]).

Now, (2.16) is equivalent to

$$\mathcal{L}(F_{\tau_m,k_n}, G_{\tau_m}) \to 0, \qquad \text{as} \quad n \to \infty,$$

for any fixed $m \in \mathbb{N}$. Since

$$V({}^{(\tau_n)}\psi_1, {}^{(\tau_n)}\psi_2, \sqrt{\tau_n}\,\sigma) \to V({}^{(c)}\psi_1, {}^{(c)}\psi_2, \sqrt{c}\,\sigma)$$

holds even almost surely by right-continuity (cf. also p. 309 in [19]), we have

$$\mathcal{L}(G_{\tau_n}, G_c) \to 0. \tag{2.17}$$

Set $d(n',n'') := \lfloor c_{n''}k_{n''}\rfloor - \lfloor \tau_{n'}k_{n''}\rfloor = \lfloor (c_{n''} - \tau_{n'})k_{n''}\rfloor + \delta$, where $|\delta| \le 1$. Now $d(n',n'')/k_{n''} \to 0$ along every pair of subsequences $\{n'\}, \{n''\} \subset \mathbb{N}$, $n', n'' \to \infty$. Thus, by an application of Lemma 2.1, along such a pair of subsequences,

$$\frac{1}{A_{k_{n''}}}\left\{\sum_{j=1}^{d(n',n'')} X_j - (c_{n''} - \tau_{n'})B_{k_{n''}}\right\} \xrightarrow{\mathbb{P}} 0.$$

From this convergence it follows that

$$\mathcal{L}(F_{\tau_{n'},k_{n''}}, F_{c_{n''}}) \to 0, \qquad \text{as} \quad n', n'' \to \infty. \tag{2.18}$$

For $n, M \in \mathbb{N}$ set $\delta(n,M) := \inf\{\mathcal{L}(F_{\tau_m,k_n}, G_{\tau_m}) : m > M\}$. We see by (2.16) that $\delta(n,M) \to 0$ for every fixed M. Thus, one can pick a subsequence $M(n)$ such that $M(n+1) \ge M(n)$, $M(n) \to \infty$, and, introducing

$$\varepsilon_n^{(1)} := \mathcal{L}(F_{c_n}, F_{\tau_{M(n)},k_n}), \quad \varepsilon_n^{(2)} := \mathcal{L}(F_{\tau_{M(n)},k_n}, G_{\tau_{M(n)}}) \quad \text{and} \quad \varepsilon_n^{(3)} := \mathcal{L}(G_{\tau_{M(n)}}, G_c),$$

we have $\varepsilon_n^{(2)} \to 0$. Setting $\{n''\} := \mathbb{N}$ and $\{n'\} := \{M(n)\}$, where $M(n) \to \infty$, from (2.18) it follows that $\varepsilon_n^{(1)} \to 0$. Finally, $\varepsilon_n^{(3)} \to 0$ by the same argument, using (2.17). Now, by the triangle inequality for the Lévy distance, all this together implies

$$\mathcal{L}(F_{c_n}, G_c) \to 0.$$

But this is just the statement to be proved. ■

Proof of Theorem 2.1. First we prove necessity; therefore we choose a subsequence $\{n'\} \subset \mathbb{N}$ such that

$$k_{n'+1}/k_{n'} \to c > 1, \qquad \text{as} \quad n' \to \infty.$$

Applying Lemma 2.3 along $\{k_{n'}\}$ with $c_{n'} = k_{n'+1}/k_{n'} \to c$, we see that

$$\frac{1}{A_{k_{n'}}} \left\{ \sum_{j=1}^{k_{n'+1}} X_j - \frac{k_{n'+1}}{k_{n'}} B_{k_{n'}} \right\} \xrightarrow{\mathcal{D}} V({}^{(c)}\psi_1, {}^{(c)}\psi_2, \sqrt{c}\,\sigma), \qquad \text{as} \quad n' \to \infty. \tag{2.19}$$

But (2.8) in itself implies that

$$\frac{1}{A_{k_{n'+1}}} \left\{ \sum_{j=1}^{k_{n'+1}} X_j - B_{k_{n'+1}} \right\} \xrightarrow{\mathcal{D}} V(\psi_1, \psi_2, \sigma), \qquad \text{as} \quad n' \to \infty. \tag{2.20}$$

Thus, by convergence of types ([44], p. 42), $A_{k_{n'+1}}/A_{k_{n'}} \to \beta \in (0, \infty)$, and so by the uniqueness of the representation of infinitely divisible variables at (2.2) we have

$${}^{(c)}\psi_1 = \beta\psi_1, \quad {}^{(c)}\psi_2 = \beta\psi_2, \quad \sqrt{c}\,\sigma = \beta\sigma. \tag{2.21}$$

Since $c > 1$ and ψ_1, ψ_2 are non-positive and non-decreasing, we see that $1 \le \beta < \infty$. We may assume that at least one of the ψ_j, $j = 1, 2$, is not identically 0 since in the opposite case (2.9) obviously holds. Then necessarily $\beta > 1$, taking into view that by $c > 1$ and the non-decreasing property of ψ_j the case $\beta = 1$ would force $\psi_j \equiv 0$, $j = 1, 2$ (from (2.1) it trivially follows that $\psi_j(s) \to 0$ as $s \to \infty$, $j = 1, 2$), and hence eliminated.

Thus, we can put $\beta = c^{1/\alpha}$ for some $\alpha \in (0, \infty)$. Introduce now $\psi_j^\alpha(s) := \psi_j(s)$, $s > 0$, $j = 1, 2$, and define $M_j(\cdot)$ by setting

$$\psi_j^\alpha(s) = -s^{-1/\alpha} M_j(s), \qquad 0 < s < \infty, \; j = 1, 2.$$

Then we have $M_j(cs) = M_j(s) \ge 0$, $0 < s < \infty$, automatically implying that M_j is bounded, $j = 1, 2$, and since $\psi_j^\alpha \in \Psi$, the functions M_j are right-continuous. Also, $M_j(s_0) = 0$ for some $s_0 \in (0, \infty)$ and $j \in \{1, 2\}$ would force $\psi_j^\alpha \equiv 0$ by the non-decreasing property. Thus, if $M_j \not\equiv 0$ then M_j is bounded away from both zero and infinity, $j = 1, 2$. The relation in (2.12) follows arguing with the non-decreasing property again and, if at least one of the ψ_j^α, $j = 1, 2$ is not identically 0 then $\alpha \in (0, 2)$ by the square integrability in (2.1). A comparison with (2.21) therefore shows that $\sigma = 0$. However, if both $\psi_1 = \psi_2 \equiv 0$, then obviously $\alpha = 2$ and $\sigma > 0$, i.e., the limiting variable in (2.8) is normal.

Turning now to sufficiency, it is enough to concentrate on the non-normal case, the statement being otherwise trivial. The properties of M_j and $0 < \alpha < 2$ make sure that $\psi_j^\alpha \in \Psi$, $j = 1, 2$. Set $k_n := \lfloor c^n \rfloor$, obviously satisfying (1.3c), and put $A_{k_n} = c^{n/\alpha}$. Then, using (1.30) in [19] and the property ${}^{(c)}\psi_j^\alpha = c^{1/\alpha}\psi_j^\alpha$, $j = 1, 2$, we have

$$\frac{1}{A_{k_n}} \sum_{j=1}^{\lfloor c^n \rfloor} V_j(\psi_1^\alpha, \psi_2^\alpha, 0) \stackrel{\mathcal{D}}{=} V\big({}^{(\lfloor c^n \rfloor / c^n)}\psi_1^\alpha, {}^{(\lfloor c^n \rfloor / c^n)}\psi_2^\alpha, 0\big) + \theta_n \tag{2.22}$$

for the independent copies $V_1(\psi_1^\alpha, \psi_2^\alpha, 0)$, $V_2(\psi_1^\alpha, \psi_2^\alpha, 0), \ldots$ of $V(\psi_1^\alpha, \psi_2^\alpha, 0)$, where the constant θ_n, recalling (2.2), is given by

$$\theta_n = \Theta\big({}^{(\lfloor c^n \rfloor / c^n)}\psi_1^\alpha\big) - c^{-n/\alpha}\Theta\big({}^{(\lfloor c^n \rfloor)}\psi_1^\alpha\big) - \Theta\big({}^{(\lfloor c^n \rfloor / c^n)}\psi_2^\alpha\big) + c^{-n/\alpha}\Theta\big({}^{(\lfloor c^n \rfloor)}\psi_2^\alpha\big).$$

Since the random variable on the right-hand side of (2.22) converges to $V(\psi_1^\alpha, \psi_2^\alpha, 0)$ even almost surely, the relation in (2.8) holds with $X_i = V_i(\psi_1^\alpha, \psi_2^\alpha, 0)$, $i = 1, 2, \ldots$, along the geometrical subsequence $k_n := \lfloor c^n \rfloor$ with $A_{k_n} \equiv c^{n/\alpha}$ and $B_{k_n} \equiv A_{k_n}\theta_n$. ■

Remark. Comparison of (2.19) and (2.20) shows that the centring and normalizing constants in (2.8), under the assumption that (1.3a) holds, necessarily satisfy

$$A_{k_{n'+1}} / A_{k_{n'}} \to c^{1/\alpha},$$

$$\frac{B_{k_{n'}} k_{n'+1}/k_{n'} - B_{k_{n'+1}}}{A_{k_{n'}}} \to -c^{1/\alpha}\Theta(\psi_1^\alpha) + \Theta({}^{(c)}\psi_1^\alpha) + c^{1/\alpha}\Theta(\psi_2^\alpha) - \Theta({}^{(c)}\psi_2^\alpha),$$

$\alpha \in (0, 2]$, where $k_{n'+1}/k_{n'} \to c$ as $n' \to \infty$ and $\psi_j^\alpha \equiv 0$ for $\alpha = 2$, $j = 1, 2$. (Recall that the centring constants B_{k_n} are such that (2.8) holds with $V(\psi_1^\alpha, \psi_2^\alpha, 0)$, instead of $V(\psi_1^\alpha, \psi_2^\alpha, 0) + d$, for some $d \in \mathbb{R}$, $d \neq 0$.)

2.3. The domains of attraction

Now we turn to the investigation of domains of attraction. We shall see that conditions (1.3b) and (1.3c) are equivalent even from the point of view of the attracted distributions.

Let us define for any semistable law $G = G_{\psi_1, \psi_2, \sigma}$, necessarily either with $\sigma > 0$ and $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ or with $(\psi_1, \psi_2, \sigma) = (\psi_1^\alpha, \psi_2^\alpha, 0)$, where at least one of ψ_1^α, ψ_2^α is not identically 0, the domain of geometric partial attraction with rank $c \geq 1$, written $\mathbb{D}_{\mathrm{gp}}^{(c)}(G)$ or $\mathbb{D}_{\mathrm{gp}}^{(c)}(\psi_1, \psi_2, \sigma)$. We say that $F \in \mathbb{D}_{\mathrm{gp}}^{(c)}(G)$ if and only if (2.8) holds along a subsequence $\{k_n\}$ satisfying (1.3c) with the given constant $c \geq 1$.

Theorem 2.2. *For a nongaussian semistable law $G_{\psi_1^\alpha,\psi_2^\alpha,0}$, define*

$$\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) := \inf\{c > 1 : M_j(cs) = M_j(s),\, 0 < s < \infty,\, j = 1,2\},$$

the minimal common period, and set $\mathbf{c} = \mathbf{c}(G_{0,0,\sigma}) := 1$ for $\sigma > 0$.

(i) *If $\mathbf{c} > 1$, then $\mathbb{D}_{\mathrm{gp}}^{(c)}(\psi_1^\alpha, \psi_2^\alpha, 0) \neq \emptyset$ if and only if $c = \mathbf{c}^{n_0}$ for some $n_0 \in \mathbb{N}$. Moreover, if (2.8) holds for some F along a subsequence $\{k_n\}$ satisfying (1.3b) with $(\psi_1, \psi_2, \sigma) = (\psi_1^\alpha, \psi_2^\alpha, 0)$, then necessarily for the constant c appearing in (1.3b) we have $c = \mathbf{c}^{l_0}$ for some $l_0 \in \mathbb{N}$, and $F \in \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c}^n)}(\psi_1^\alpha, \psi_2^\alpha, 0)$ for every $n \in \mathbb{N}$.*

(ii) *In case of $\mathbf{c} = 1$, if (2.8) holds for some F along a subsequence $\{k_n\}$ satisfying (1.3b) with $(\psi_1, \psi_2, \sigma) = (\psi_1^\alpha, \psi_2^\alpha, 0)$ or with $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$, then F belongs to the domain of attraction of $G_{\psi_1,\psi_2,\sigma}$, i.e., $F \in \mathbb{D}(G_{\psi_1,\psi_2,\sigma})$, and hence $G_{\psi_1,\psi_2,\sigma}$ is necessarily stable; thus either $\sigma > 0$ and $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ or $(\psi_1, \psi_2, \sigma) = (m_1\psi^\alpha, m_2\psi^\alpha, 0)$, where $\alpha \in (0, 2)$, $\psi^\alpha(s) = -s^{-1/\alpha}$, $s > 0$, $m_1, m_2 \geq 0$, $m_1 + m_2 > 0$, and, for suitable $A_n > 0$ and B_n, (2.8) holds with $\{k_n\} = \mathbb{N}$.*

We know from the proof of the sufficiency part of Theorem 2.1 that $G \in \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(G)$, for every semistable G, which will also be one of the first steps in the proof of the present theorem. Here part (i) clearly implies that the set of distributions attracted in geometrical order to some nonstable semistable law G coincides with $\mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(G)$. Thus, it makes sense to define the domain of geometric partial attraction

$$\mathbb{D}_{\mathrm{gp}}(G) := \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(G) = \bigcup_{c \geq 1} \mathbb{D}_{\mathrm{gp}}^{(c)}(G) = \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c}^n)}(G),$$

for every $n \in \mathbb{N}$. For a stable law S with exponent $\alpha \in (0, 2]$ set $\mathbb{D}_{\mathrm{gp}}(S) := \bigcup_{c \geq 1} \mathbb{D}_{\mathrm{gp}}^{(c)}(S)$. Let us spell out the obvious consequence of Theorem 2.2:

Corollary 2.2. *For a stable law $S \in \mathcal{S}$ with exponent $\alpha \in (0, 2]$, $\mathbb{D}_{\mathrm{gp}}(S) = \mathbb{D}(S)$, i.e., the domain of geometric partial attraction and the domain of attraction coincide.* ■

We turn now to the characterization of the attracted distributions. Since $\mathbb{D}_{\mathrm{gp}}(2) = \mathbb{D}(2)$, and $\mathbb{D}(2)$ has been characterized in the classical literature, see e.g. Corollary 1 in [26] and the ensuing discussion, we do not need to consider this case. Let $\{k_n\}_{n=0}^\infty$ be a subsequence of $\mathbb{N}$ satisfying (1.3c), which is assumed to be unbounded and eventually strictly increasing. Then for every $s \in (0, 1)$ small enough there exists a uniquely determined $k_{n^*(s)}$ such that $1/k_{n^*(s)} \leq s < 1/k_{n^*(s)-1}$. Set $\gamma(s) := s k_{n^*(s)}$; in case of $k_{n+1}/k_n \to 1$ we simply put $\gamma(s) \equiv 1$. Then, by (1.3c), for any fixed $\varepsilon > 0$ and s small enough, $1 \leq \gamma(s) < c + \varepsilon$, with the same c as in (1.3c). Thus, for any sequence $s_m > 0$, $s_m \to 0$, $m \to \infty$, the limit points of the sequence $\{\gamma(s_m)\}$ are in the closed interval $[1, c]$.

Theorem 2.3. (Domain theorem) (i) *Suppose that* $F \in \mathbb{D}_{\mathrm{gp}}(\psi_1^\alpha, \psi_2^\alpha, 0)$ *along a subsequence* $\{k_n\}$ *satisfying (1.3c) for a nongaussian semistable law* $G_{\psi_1^\alpha,\psi_2^\alpha,0}$, *with* ψ_1^α, ψ_2^α *as in (2.10). Then for all* $s \in (0,1)$ *small enough the quantile function* Q *pertaining to* F *is necessarily of the form:*

$$\begin{aligned} Q_+(s) &= -s^{-1/\alpha}\ell(s)\big[M_1(\gamma(s)) + h_1(s)\big], \\ Q(1-s) &= s^{-1/\alpha}\ell(s)\big[M_2(\gamma(s)) + h_2(s)\big], \end{aligned} \tag{2.23}$$

where $0 < \alpha < 2$, ℓ *is right-continuous and slowly varying at 0, and* h_1 *and* h_2 *are some right-continuous functions such that*

$$h_j(s/k_n) \to 0 \qquad \text{as} \quad n \to \infty, \tag{2.24}$$

for every continuity point s *of* M_j, *j=1,2. If, for some* $j \in \{1,2\}$, M_j *is continuous, then (2.24) can be further specified as*

$$h_j(s) \to 0 \qquad \text{as} \quad s \to 0. \tag{2.25}$$

Conversely, given a subsequence $\{k_n\}$ *satisfying (1.3c) and the corresponding function* $\gamma(\cdot)$, *suppose that (2.23) and (2.24) are satisfied with* $0 < \alpha < 2$, h_1, h_2 *and* ℓ *right-continuous, and* ℓ *also slowly varying at 0. Then for any fixed pair of integers* $l, m \geq 0$,

$$\frac{1}{k_n^{1/\alpha}\ell(1/k_n)} \left\{ \sum_{j=l+1}^{k_n-m} X_{j,k_n} - k_n \mu_{l,m}(k_n) \right\} \xrightarrow{\mathcal{D}} V_{l,m}(\psi_1^\alpha, \psi_2^\alpha, 0), \tag{2.26}$$

where $\mu_{l,m}(n)$ *is as in (2.6) and the limiting random variable is as in (2.2). In particular, for the distribution function* F *pertaining to* Q *we have* $F \in \mathbb{D}_{\mathrm{gp}}(\psi_1^\alpha, \psi_2^\alpha, 0)$ *along the given subsequence* $\{k_n\}_{n=0}^\infty$.

(ii) *Suppose that* $F \in \mathbb{D}_{\mathrm{gp}}(\psi_1^\alpha, \psi_2^\alpha, 0)$ *along the subsequence* $k_n = \lfloor c^n \rfloor$, *where* $c > 1$ *is necessarily a common period of the functions* M_j, $j = 1, 2$. *Then for all* $s \in (0,1)$ *small enough the quantile function* Q *pertaining to* F *may be written as:*

$$\begin{aligned} Q_+(s) &= -s^{-1/\alpha}\ell(s)\big[M_1(s) + h_1(s)\big], \\ Q(1-s) &= s^{-1/\alpha}\ell(s)\big[M_2(s) + h_2(s)\big], \end{aligned} \tag{2.27}$$

with α, ℓ, h_1 *and* h_2 *having the properties listed above, i.e., we have (2.23) with* $\gamma(s) \equiv s$. *Conversely, if (2.27) is satisfied then (2.26) holds true with* $k_n = \lfloor c^n \rfloor$ *for every* $c > 1$ *which is a common period of the functions* M_j, $j = 1, 2$.

Theorem 2.3 completely describes the domain of geometric partial attraction of a nonnormal semistable law. Note that the normalizing constants appearing in (2.26) are the same as in the stable case, evaluated along $\{k_n\}$. In the discontinuous case $\limsup_{s\downarrow 0} |h_j(s)| > 0$ may indeed happen; consider e.g. the following quantile function:

$$Q(1-u) := u^{-1}\big[2^{\lceil \mathrm{Log}(1/u)\rceil - \mathrm{Log}(1/u)} - (\lceil \mathrm{Log}(1/u)\rceil - \mathrm{Log}(1/u))^{\lceil \mathrm{Log}(1/u)\rceil}\big], \qquad (2.28)$$

$0 < u < 1$, where and in the sequel Log stands for the logarithm to the base 2. It is straightforward to check that the function thus defined is monotone increasing on $(0,1)$. Setting $\alpha = 1$ and $M_2(u) := 2^{\lceil \mathrm{Log}(1/u)\rceil - \mathrm{Log}(1/u)}$, we have $M_2(u) = M_2(2u)$, $u \in (0,\infty)$. Choosing $k_n = 2^n$, we have the representation in (2.27) with $M_1 \equiv 0$, $h_1(u) = -2u$ and $h_2(u) = -(\lceil \mathrm{Log}(1/u)\rceil - \mathrm{Log}(1/u))^{\lceil \mathrm{Log}(1/u)\rceil}$. In fact, this is also the representation in (2.23), since for the function $\gamma(\cdot)$ determined by $k_n = 2^n$ we have $M_j(\gamma(u)) = M_j(u)$ for each $u \in (0,1)$, $j = 1,2$, c being an integer. Now $h_2(s/2^n) \to 0$ for every continuity point of M_2, i.e., for $s \in (0,\infty) \setminus \{2^j : j \in \mathbb{Z}\}$, but $\limsup_{s\downarrow 0} |h_2(s)| = 1$.

Let us now formulate the result in Theorem 2.3 for distribution functions, retaining the notation for the ingredients of Lévy measures introduced in the discussion of Theorem 2.1 and setting $A_n := n^{1/\alpha}\ell(1/n)$. Obviously $A_{k_{n+1}}/A_{k_n} \to c^{1/\alpha}$ and so for $c > 1$ and all $x \in \mathbb{R}$ large enough there exists a uniquely determined A_{k_n}, which will be denoted by $a(x)$, such that $A_{k_n} \le x < A_{k_{n+1}}$. In case of $c > 1$ we represent x as $x = \delta(x)a(x)$ and we set $\delta(x) \equiv 1$ if $c = 1$. For every fixed $\varepsilon > 0$ we have $\delta(x) \in [1, c^{1/\alpha} + \varepsilon]$ for all x large enough.

Corollary 2.3. *$F \in \mathbb{D}_{\mathrm{gp}}(\psi_1^\alpha, \psi_2^\alpha, 0)$ along the subsequence k_n with normalizing constants A_{k_n} if and only if for all $x \in \mathbb{R}$ large enough*

$$\begin{aligned} F_-(-x) &= x^{-\alpha}\ell^*(x)\big[M_L(-\delta(x)) + h_L(x)\big], \\ 1 - F(x) &= x^{-\alpha}\ell^*(x)\big[M_R(\delta(x)) + h_R(x)\big], \end{aligned} \qquad (2.29)$$

where ℓ^* is a right-continuous function, slowly varying at ∞, defined by

$$x^{-\alpha}\ell^*(x) := \sup\{t : t^{-1/\alpha}\ell(t) > x\}, \qquad x > 0, \qquad (2.30)$$

$\alpha \in (0,2)$, F_- is the left-continuous version of F and the right-continuous error functions h_R and h_L are such that

$$h_K(A_{k_n}x_0) \to 0 \qquad \text{as} \quad n \to \infty, \qquad (2.31)$$

for every continuity point x_0 of M_K, $K \in \{L, R\}$. If, for $K \in \{L, R\}$, M_K is continuous then the relation in (2.31) can be further strengthened to

$$h_K(x) \to 0 \qquad \text{as} \quad x \to \infty. \qquad (2.32)$$

The case of the geometrical subsequence $k_n = \lfloor c^n \rfloor$, $c > 1$, can be handled similarly as in part (ii) of Theorem 2.3, replacing $\delta(x)$ by x. The representation for F given here describes the attracted distributions along previously fixed sequences of k_n and A_{k_n}. If one is interested in describing the attracted distributions fixing only k_n then this can be done by the free choice of ℓ. If one fixes only the normalizing sequence A'_n such that $A'_{n+1}/A'_n \to c^{1/\alpha}$, then the convergence in (2.8) may happen along arbitrary subsequence k_n such that $k_{n+1}/k_n \to c$ and $A_{k_n} = A'_n$. In this case ℓ can be defined by $\ell(s) := A'_n/k_n^{1/\alpha}$, $s \in [1/k_n, 1/k_{n-1})$, and up to asymptotic equivalence any slowly varying function ℓ^* may appear in (2.29). This last version of our characterization is mathematically equivalent with the one given in Grinevich and Khoklov [49], after correcting an oversight there. They claim $h_K(x) \to 0$ as $x \to \infty$, $K \in \{L, R\}$, even in the discontinuous case, which is not necessarily true, as it can be shown by a counterexample constructed in the manner of (2.28). The seemingly little difference has substantial consequences in [32] and [33]. We believe that our characterization, which is in terms of the subsequence $\{k_n\}$ is more natural than theirs, happening in terms of the normalizing sequence. We state now a further consequence of Theorem 2.3.

Corollary 2.4. *Let $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ be a nongaussian semistable law, necessarily with $\alpha \in (0,2)$, and suppose that $F \in \mathbb{D}_{\mathrm{gp}}(\psi_1^\alpha, \psi_2^\alpha, 0)$. Then*

(i) *$G_{\psi_1^\alpha,\psi_2^\alpha,0}$ has finite absolute moments of order $\delta \in (0,\alpha)$ and infinite moments of order $\eta \geq \alpha$;*

(ii) *F has finite absolute moments of order $\delta \in (0,\alpha)$ and infinite moments of order $\eta \geq \alpha$.* ■

The first statement readily follows from Theorem 2.1, and part (ii) can be obtained from Theorem 2.3, using the boundedness of the functions $M_1 + h_1$ and $M_2 + h_2$ (cf. (2.41) and the proof of the sufficiency part of Theorem 2.3).

In a fashion as done in the stable case, one can define the domain of normal geometric partial attraction of a nongaussian semistable law $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ as the subset of $\mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ consisting of distribution functions for which the convergence in (2.8) takes place along a subsequence $\{k_n\}$ satisfying (1.3c) and, in addition, the normalizing constants can be chosen as $A_{k_n} = k_n^{1/\alpha}$. By Theorem 2.3 it is clear that this happens for an F if and only if $\ell(s) \to 1$, as $s \to 0$, in the representation at (2.23)

We have seen that conditions (1.3b) and (1.3c) are practically the same as long as we consider domains of geometric partial attraction. However, it will turn out that there is an essential difference between these two conditions and the one at (1.3a). Although the limiting distributions of sums of independent identically distributed random variables along subsequences satisfying (1.3a) are necessarily semistable, the class of distributions partially attracted to some semistable distribution along such a subsequence is not narrower than the class of distributions attracted to it along *any* subsequence. Note that $\mathbb{D}_{\mathrm{gp}}(G)$ is defined through (1.3c) and *not* through (1.3a).

Theorem 2.4. *For a given semistable law G, let* c *be as in Theorem 2.2.*

(i) *The class of distributions attracted to G along a subsequence $\{k_n\}$ satisfying (1.3a) (necessarily with $c = \mathbf{c}^{n_0}$ for some $n_0 \in \mathbb{N}$ in case of $\mathbf{c} > 1$) coincides with the whole class $\mathbb{D}_{\mathrm{p}}(G)$. Moreover, for a nongaussian semistable law $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$, if $F \in \mathbb{D}_{\mathrm{p}}(\psi_1^\alpha, \psi_2^\alpha, 0)$ then necessarily there exists some subsequence $\{n'\} \in \mathbb{N}$ such that*

$$\psi_j(n'; s) \Longrightarrow \beta\psi_j^\alpha(s), \qquad 0 < s < \infty, \quad j = 1, 2, \tag{2.33}$$

for some $\beta \in (0, \infty)$, where the functions $\psi_j(n', s)$ are as in (2.7), and

$$\lim_{h\to\infty} \limsup_{n'\to\infty} \frac{\sigma(h/n')}{\sigma(1/n')} = 0. \tag{2.34}$$

Conversely, if the above conditions are satisfied then for any fixed pair of integers $l, m \geq 0$ we have

$$\frac{1}{a(n')} \left\{ \sum_{j=l+1}^{n'-m} X_{j,n'} - n'\mu_{l,m}(n') \right\} \xrightarrow{\mathcal{D}} \beta V_{l,m}(\psi_1^\alpha, \psi_2^\alpha, 0) \qquad \text{as} \quad n' \to \infty. \tag{2.35}$$

(ii) *For any nonnormal semistable law $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$, $\mathbb{D}_{\mathrm{gp}}(G)$ is a proper subset of $\mathbb{D}_{\mathrm{p}}(G)$.*

Proofs

Proof of Theorem 2.2. The first statement of part (i) is almost obvious. Suppose that $F \in \mathbb{D}_{\mathrm{gp}}^{(c')}(\psi_1^\alpha, \psi_2^\alpha, 0)$ for some $c' \geq 1$ along some subsequence $\{k_n'\}$. If $c' = 1$ then for any given c'' one can choose a further subsequence $\{k_n''\} \subset \{k_n'\}$ such that $k_{n+1}''/k_n'' \to c''$, which means by Theorem 2.1 that c'' is also a common period of M_1 and M_2. Since c'' was arbitrary, this forces $\mathbf{c} = 1$, and hence eliminated. If $c' > 1$ then by Theorem 2.1 again we see that c' is a common period, but then, by the minimality of $\mathbf{c}$, it is easy to see that $c' = \mathbf{c}^{n_0}$ for some $n_0 \in \mathbb{N}$. The consideration in the sufficiency part of the preceding theorem shows that that $\mathbb{D}_{\mathrm{gp}}^{(\mathbf{c}^n)}(\psi_1^\alpha, \psi_2^\alpha, 0) \neq \emptyset$ for $n \in \mathbb{N}$ since $V(\psi_1^\alpha, \psi_2^\alpha, 0) \in \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(\psi_1^\alpha, \psi_2^\alpha, 0)$ and so $V(\psi_1^\alpha, \psi_2^\alpha, 0) \in \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c}^n)}(\psi_1^\alpha, \psi_2^\alpha, 0)$ as well.

It remains to show that if (2.8) holds along a subsequence $\{k_n\}$ satisfying (1.3b), then $F \in \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c}^n)}(\psi_1^\alpha, \psi_2^\alpha, 0)$ for every $n \in \mathbb{N}$. To this aim it is clearly enough to show that $F \in \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(\psi_1^\alpha, \psi_2^\alpha, 0)$. It can be seen in the same way as above that $c = \mathbf{c}^{n_0}$ for the c appearing in (1.3b) with some $n_0 \in \mathbb{N}$. Define the following sequences for $0 \leq j \leq n_0 - 1$:

$$k_n^{(j)} := \lfloor \mathbf{c}^j k_n \rfloor, \qquad A_{k_n^{(j)}} := \mathbf{c}^{j/\alpha} A_{k_n}, \qquad B_{k_n^{(j)}} := \mathbf{c}^j B_{k_n} + A_{k_n^{(j)}} \theta_{k_n^{(j)}},$$

where

$$\theta_{k_n^{(j)}} = \Theta(\psi_1^\alpha) - \mathbf{c}^{-j/\alpha}\Theta\big({}^{(\mathbf{c}^j)}\psi_1^\alpha\big) - \Theta(\psi_2^\alpha) + \mathbf{c}^{-j/\alpha}\Theta\big({}^{(\mathbf{c}^j)}\psi_2^\alpha\big).$$

Then, by Lemma 2.3, (2.8) holds along each subsequence $\{k_n^{(j)}\}$, $0 \le j \le n_0 - 1$. Thus, defining the union sequence $\{l_n\}_{n=1}^{\infty} := \bigcup_{j=0}^{n_0-1} \{k_n^{(j)}\}_{n=1}^{\infty}$, $l_{n+1} > l_n$, (2.8) holds along $\{l_n\}$ just as well. For the sequence $\{l_n\}$ we have

$$\limsup_{n\to\infty} l_{n+1}/l_n = \mathbf{c}. \tag{2.36}$$

Now we define a subsequence $\{r_n\} \subset \{l_n\}$. Let $r_1 = l_1$ and then put $r_n = \min\{l_j : l_j \ge \mathbf{c}^{1/4} r_{n-1}\}$. By (2.36), the set $\{n \in \mathbb{N} : r_{n+1}/r_n \ge \mathbf{c}^{3/2}\}$ is finite. Thus we have

$$\mathbf{c}^{1/4} \le \liminf_{n\to\infty} r_{n+1}/r_n \le \limsup_{n\to\infty} r_{n+1}/r_n \le \mathbf{c}^{3/2}. \tag{2.37}$$

Suppose now that for some subsequences $\{n_1(m)\}, \{n_2(m)\} \subset \mathbb{N}$, $n_1(m) > n_2(m)$, $m = 1, 2, \dots$, the limit $\lim_{m\to\infty} l_{n_1(m)}/l_{n_2(m)} = c' > 1$ exists and finite. Arguing in the same way as in the proof of Theorem 2.1 we obtain that c' is a common period, and so $c' = \mathbf{c}^{n_0}$ for some $n_0 \in \mathbb{N}$, necessarily. Applying this observation to (2.37), it follows that

$$\liminf_{n\to\infty} r_{n+1}/r_n = \limsup_{n\to\infty} r_{n+1}/r_n = \mathbf{c}.$$

Thus $F \in \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(\psi_1^\alpha, \psi_2^\alpha, 0)$ along $\{r_n\}$, as claimed.

Turning now to part (ii), for the sake of a unified notation introduce

$$(\psi_1, \psi_2, \sigma) := \begin{cases} (\psi_1^\alpha, \psi_2^\alpha, 0), & \alpha \in (0,2), \\ (0, 0, \sigma), & \alpha = 2. \end{cases}$$

Observe that $\mathbf{c} = 1$ implies that the functions M_j, $j = 1, 2$, are constant, thus the limiting random variable in (2.8) is indeed stable, since either $\sigma > 0$ and $(\psi_1, \psi_2, \sigma) = (0, 0, \sigma)$ or $(\psi_1, \psi_2, \sigma) = (m_1 \psi^\alpha, m_2 \psi^\alpha, 0)$, where $\alpha \in (0,2)$, $\psi^\alpha(s) = -s^{-1/\alpha}$, $s > 0$, $m_1, m_2 \ge 0$, $m_1 + m_2 > 0$. In particular, ${}^{(\lambda)}\psi_j = \lambda^{1/\alpha}\psi_j$, $j = 1, 2$, for each $\lambda > 0$. Introduce for $l < m$, $l, m \in \mathbb{N} \cup \{0\}$ the following sequences:

$$k_n^{(l/m)} := \lfloor c^{l/m} k_n \rfloor, \qquad A_{k_n^{(l/m)}} := A_{k_n} c^{l/m\alpha}, \qquad B_{k_n^{(l/m)}} := B_{k_n} c^{l/m} + A_{k_n^{(l/m)}} \theta_{k_n^{(l/m)}},$$

where $\{k_n\}$ is the subsequence satisfying (1.3b) figuring in (2.8), c is as in (1.3b) and

$$\theta_{k_n^{(l/m)}} = \Theta(\psi_1) - c^{-l/m\alpha}\Theta\big({}^{(c^{l/m})}\psi_1\big) - \Theta(\psi_2) + c^{-l/m\alpha}\Theta\big({}^{(c^{l/m})}\psi_2\big).$$

Clearly, by Lemma 2.3 and the fact that ${}^{(c^{l/m})}\psi_j = c^{l/m\alpha}\psi_j$, $j = 1, 2$,

$$\frac{1}{A_{k_n^{(l/m)}}} \left\{ \sum_{j=1}^{k_n^{(l/m)}} X_j - B_{k_n^{(l/m)}} \right\} \xrightarrow{\mathcal{D}} V(\psi_1, \psi_2, \sigma), \tag{2.38}$$

for any fixed pair l, m, or, what is the same, we have $L(F_n^{(l/m)}, G_{\psi_1,\psi_2,\sigma}) \to 0$, where $F_n^{(l/m)}$ denotes the distribution functions of the random variables on the left-hand side of (2.38). Let now $N(m)$, $m \in \mathbb{N}$, be a sequence of integers such that $N(m+1) > N(m)$ and $L(F_n^{(l/m)}, G_{\psi_1,\psi_2,\sigma}) \leq 1/m$, $0 \leq l \leq m$, for all $n \geq N(m)$. Define the union sequence $\{r_m\}_{m=0}^{\infty} := \bigcup_{m=1}^{\infty} \bigcup_{l=0}^{m-1} \bigcup_{n=N(m)}^{N(m+1)} \{k_n^{(l/m)}\}$, $r_{m+1} > r_m$. By the definition of $\{r_m\}$,

$$\limsup_{m\to\infty} r_{m+1}/r_m = 1 \tag{2.39}$$

and

$$\frac{1}{A_{r_m}} \left\{ \sum_{j=1}^{r_m} X_j - B_{r_m} \right\} \xrightarrow{\mathcal{D}} V(\psi_1, \psi_2, \sigma), \qquad \text{as} \quad m \to \infty. \tag{2.40}$$

For every $n \in \mathbb{N}$ set $R(n) := \max\{r_m : r_m \leq n\}$ and let $A_n = A_{R(n)}$ and $B_n = \frac{n}{R(n)} B_{R(n)}$. By (2.39), $\lim_{n\to\infty} (n - R(n))/R(n) = 0$. Thus, by an application of Lemma 2.1, it follows from (2.40) that

$$\frac{1}{A_n} \left\{ \sum_{j=1}^{n} X_j - B_n \right\} \xrightarrow{\mathcal{D}} V(\psi_1, \psi_2, \sigma),$$

i.e., we have $F \in \mathbb{D}(\psi_1, \psi_2, \sigma)$, as claimed. ∎

Remark. Note that if for some $c', c'' > 1$ $\mathbb{D}_{\mathrm{gp}}^{(c'')}(G) \neq \emptyset$, $\mathbb{D}_{\mathrm{gp}}^{(c')}(G) \neq \emptyset$ and $\log c' / \log c''$ is irrational, then G is necessarily stable.

To prove Theorem 2.3 first we separate a lemma, which will play a crucial role in Section 3, too.

Lemma 2.4. *If a quantile function $Q(\cdot)$ satisfies*

$$\begin{gathered} C_1 s^{-1/\alpha} \ell(s) < |Q_+(s)| < D_1 s^{-1/\alpha} \ell(s) \\ \text{and} \\ C_2 s^{-1/\alpha} \ell(s) < |Q(1-s)| < D_2 s^{-1/\alpha} \ell(s) \end{gathered} \tag{2.41}$$

for all $s > 0$ sufficiently small, where $0 \leq C_1 < D_1 < \infty$ and $0 \leq C_2 < D_2 < \infty$ are constants such that $C_1 + C_2 > 0$ and $C_j = 0$ if and only if $D_j > 0$ can be chosen as small as we wish, then for the function $\sigma(s,t)$ given by (2.4) we have:

(i) *There exist some constants $K_1, K_2 \in (0, \infty)$ such that*

$$K_1 \leq \liminf_{s\downarrow 0} \frac{\sigma^2(s, 1-s)}{s^{1-\frac{2}{\alpha}} \ell^2(s)} \leq \limsup_{s\downarrow 0} \frac{\sigma^2(s, 1-s)}{s^{1-\frac{2}{\alpha}} \ell^2(s)} \leq K_2;$$

(ii) *If $C_1 > 0$ in (2.41), then there exist some constants $K_1^{(1)}, K_2^{(1)} \in (0, \infty)$ such that*

$$K_1^{(1)} \leq \liminf_{s\downarrow 0} \frac{\sigma^2(s, 1/2)}{s^{1-\frac{2}{\alpha}} \ell^2(s)} \leq \limsup_{s\downarrow 0} \frac{\sigma^2(s, 1/2)}{s^{1-\frac{2}{\alpha}} \ell^2(s)} \leq K_2^{(1)},$$

and if $C_2 > 0$ in (2.41), then there exist some constants $K_1^{(2)}, K_2^{(2)} \in (0,\infty)$ such that

$$K_1^{(2)} \le \liminf_{s\downarrow 0} \frac{\sigma^2(1/2\,,1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{\sigma^2(1/2\,,1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le K_2^{(2)}.$$

Proof. We follow the proof of Lemma 1 in [30]. Obviously the inequalities in (2.41) directly imply that

$$C_1^2 + C_2^2 \le \liminf_{s\downarrow 0} \frac{sQ^2(s) + sQ^2(1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{sQ^2(s) + sQ^2(1-s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le D_1^2 + D_2^2$$

and

$$\lim_{s\downarrow 0} \frac{s|Q(s)| + s|Q(1-s)|}{s^{\frac{1}{2}-\frac{1}{\alpha}}\ell(s)} = 0.$$

Also from (2.41), similarly as at (3.11) and (3.12) in [30],

$$K_1^* \le \liminf_{s\downarrow 0} \frac{\int_s^{1-s} Q^2(u)\,du}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{\int_s^{1-s} Q^2(u)\,du}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le K_2^* \tag{2.42}$$

for some constants $K_1^*, K_2^* \in (0,\infty)$ and

$$\lim_{s\downarrow 0} \frac{\int_s^{1-s} |Q(u)|\,du}{s^{\frac{1}{2}-\frac{1}{\alpha}}\ell(s)} = 0. \tag{2.43}$$

Using these four relations in the second formula in (2.4), the inequalities in (i) follow.

The symmetric two statements in (ii) are obtained in a similar fashion. Considering the first, for example, the first pair of inequalities in (2.41) imply that

$$C_1^2 \le \liminf_{s\downarrow 0} \frac{sQ^2(s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le \limsup_{s\downarrow 0} \frac{sQ^2(s)}{s^{1-\frac{2}{\alpha}}\ell^2(s)} \le D_1^2 \quad \text{and} \quad \lim_{s\downarrow 0} \frac{s|Q(s)|}{s^{\frac{1}{2}-\frac{1}{\alpha}}\ell(s)} = 0,$$

and (2.42) and (2.43) remain true by the same argument if $1-s$ in the upper limits of the integrals is replaced by $1/2$. ■

Proof of Theorem 2.3. First we deal with part (i). Sufficiency is trivial if the c appearing in (1.3c) equals 1. For $c > 1$ set

$$C(s) := \frac{s}{\max\{c^j : c^j \le s,\, j \in \mathbb{Z}\}}, \qquad 0 < s < \infty.$$

Clearly, $1 \le C(s) < c$ and $M_j(s) = M_j(C(s))$, $j = 1, 2$. Since $k_{n+1}/k_n \to c$, we have $\lim_{n\to\infty} \gamma(s/k_n) = \lim_{n\to\infty} \gamma(C(s)/k_n) = C(s)$, $0 < s < \infty$, with the exception of the case $C(s) = 1$, when the sequence $\gamma(s/k_n)$ may have two limit points, 1 and c. But in both cases $\lim_{n\to\infty} M_j(\gamma(s/k_n)) = \lim_{n\to\infty} M_j(\gamma(C(s)/k_n)) = M_j(C(s)) = M_j(s)$ at every continuity point s of M_j, $j=1,2$. Thus, using (2.24), for a Q given by (2.23) we have

$$\frac{Q_+(\frac{s}{k_n})}{A_{k_n}} \Longrightarrow -s^{-1/\alpha}M_1(s)=\psi_1^\alpha(s) \quad \text{and} \quad \frac{-Q(1-\frac{s}{k_n})}{A_{k_n}} \Longrightarrow -s^{-1/\alpha}M_2(s)=\psi_2^\alpha(s),$$

where $A_{k_n} := k_n^{1/\alpha}\ell(1/k_n)$. Since at least one of ψ_j^α, $j=1,2$, is not zero in $(1,\infty)$, by Theorem 6 and Theorem 1* of [19] it remains to prove only that the arising limiting distribution has no normal component. To do this it suffices to show by the same theorems from [19] that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{\sigma(h/k_n)}{\sigma(1/k_n)}=0. \tag{2.44}$$

First we claim that for the quantile function given by (2.23) the conditions of Lemma 2.4 are satisfied: the case $C_j=0$ may obviously only happen if the corresponding $M_j(\cdot)$ is 0. In the continuous case the claim is trivial; but if at least one of the M_j has jumps, some consideration is needed. Assume, to the contrary, that for $j\in\{1,2\}$ there is no such C_j. Then there exists a sequence $\{s_m\}$, $s_m\downarrow 0$, such that

$$\lim_{m\to\infty}\left[M_j(\gamma(s_m))+h_j(s_m)\right]\to C'\le 0.$$

Pick some $\varepsilon>0$ such that $c+\varepsilon$ is a continuity point of M_j; as it has already been noted, $\gamma(s_m)\ge c+\varepsilon$ may happen only a finite number of times. Introduce also $\gamma_m:=\gamma(s_m)$ and $k_m:=k_{n^*(s_m)}$, where $n^*(\cdot)$ is defined right above the statement of Theorem 2.3. Then we have

$$\begin{aligned}
C' &= \lim_{m\to\infty}\frac{(\gamma_m/k_m)^{-1/\alpha}\,\ell(\gamma_m/k_m)\left[M_j(\gamma_m)+h_j(\gamma_m/k_m)\right]}{(\gamma_m/k_m)^{-1/\alpha}\,\ell(\gamma_m/k_m)}\\
&\ge \limsup_{m\to\infty}\frac{((c+\varepsilon)/k_m)^{-1/\alpha}\,\ell((c+\varepsilon)/k_m)\left[M_j(c+\varepsilon)+h_j((c+\varepsilon)/k_m)\right]}{(1/k_m)^{-1/\alpha}\,\ell(\gamma_m/k_m)}\\
&=(c+\varepsilon)^{-1/\alpha}M_j(c+\varepsilon),
\end{aligned}$$

where the inequality holds since the numerator of the left side is non-increasing (being $|Q(s+)|$ or $Q(1-s)$) and the equality holds by (2.24) and the slow variation of ℓ, using also that $\gamma_m\in[1,c+\varepsilon]$ for all m large enough. Since $(c+\varepsilon)^{-1/\alpha}M_j(c+\varepsilon)$ is positive if $M_j\not\equiv 0$, there exists some $C_j>0$ if and only if $M_j\not\equiv 0$, as claimed. However, if $M_j\equiv 0$ then (2.41) is satisfied with $C_j=0$ and $D_j>0$ arbitrarily small. The existence of a $D_j<\infty$ in the general case can be proved in a similar fashion. The statement in (2.44) now follows by Lemma 2.4.

Now we turn to necessity. Fix an $s_0\in(0,\infty)$ which is a continuity point of both M_j. Introduce the right-continuous function

$$\ell(s):=\frac{s_0^{1/\alpha}}{M_1(s_0)+M_2(s_0)}\left[Q\left(1-\frac{s_0}{k_{n^*(s)}}\right)-Q_+\left(\frac{s_0}{k_{n^*(s)}}\right)\right]k_{n^*(s)}^{-1/\alpha},$$

which is positive for all s small enough. We claim that ℓ is slowly varying at zero.

The assumption that (2.8) holds with $\psi_j = \psi_j^\alpha$, $j = 1, 2$, and $\sigma = 0$ forces

$$\frac{Q_+(\frac{s}{k_n})}{A_{k_n}} \Longrightarrow -s^{-1/\alpha} M_1(s) \quad \text{and} \quad \frac{-Q(1-\frac{s}{k_n})}{A_{k_n}} \Longrightarrow -s^{-1/\alpha} M_2(s), \tag{2.45}$$

by an application of Theorem 6 (ii) of [19] and the uniqueness of the representation at (2.2). From the remark after the proof of Theorem 2.1 it follows that for any fixed $l \in \mathbb{N}$,

$$\left(\frac{k_{n+l}}{k_n}\right)^{-1/\alpha} \left(\frac{A_{k_{n+l}}}{A_{k_n}}\right) \to 1. \tag{2.46}$$

For $\lambda, s \in (0, \infty)$, define $T(\lambda, s) \in \mathbb{Z}$ by $T(\lambda, s) := n^*(\lambda s) - n^*(s)$. The key observation is that $T(\lambda, s)$ is bounded as $s \downarrow 0$ for λ fixed, therefore from (2.46) it follows that

$$\left(\frac{k_{n^*(\lambda s)}}{k_{n^*(s)}}\right)^{-1/\alpha} \left(\frac{A_{k_{n^*(\lambda s)}}}{A_{k_{n^*(s)}}}\right) \to 1. \tag{2.47}$$

But (2.45) and (2.47) together imply that $\lim_{s\downarrow 0} \ell(\lambda s)/\ell(s) = 1$, i.e., ℓ is indeed slowly varying.

Now let $s > 0$ be an arbitrary continuity point of M_1 and introduce $s_n := s/k_n$. First we consider the case $M_1 \not\equiv 0$. Then, by (2.45) and (2.47),

$$\begin{aligned}
&\frac{-Q_+(s_n)}{s_n^{-1/\alpha} \ell(s_n) M_1(\gamma(s_n))} \\
&\quad = \frac{-s_0^{-1/\alpha} s_n^{1/\alpha} k_{n^*(s_n)}^{1/\alpha} [M_1(s_0) + M_2(s_0)] Q_+(s_n) A_{k_{n^*(s_n)}}}{\left[Q\left(1 - \frac{s_0}{k_{n^*(s_n)}}\right) - Q_+\left(\frac{s_0}{k_{n^*(s_n)}}\right)\right] M_1(\gamma(s_n)) A_{k_n}} \cdot \frac{A_{k_n}}{A_{k_{n^*(s_n)}}} \to 1,
\end{aligned}$$

as $n \to \infty$, using that $k_n = k_{n^*(1/k_n)}$. (Recall also that $M_1(\gamma(s/k_n)) \to M_1(s)$ at any continuity point s.) If $M_1 \equiv 0$, a similar, but easier consideration applies. Thus, for $j = 1$ we have the representation in (2.23) together with (2.24); the other side, i.e., the case $j = 2$, can be handled similarly. The statement in (2.25) now follows by an application of the uniform convergence theorem of slowly varying functions and from the fact that the first or second convergence statement in (2.45) holds uniformly in any closed subinterval of $(0, \infty)$ if M_1 or M_2 is continuous, respectively, Q being monotone.

The statements in part (ii) now follow by a straightforward modification of the considerations above. ■

Proof of Corollary 2.3. It follows from an application of Theorem 1.5.12 in [13] that ℓ^* defined by (2.30) is indeed slowly varying. Although the whole statement follows by an asymptotic inversion from (2.23), the 'brute force' approach is rather cumbersome.

Alternatively, sufficiency can be seen in the same way as in [49], implementing an analogue of (2.41) and reducing to the stable case. Regarding the necessity part, define $B^*(\cdot)$ by

$$x^{-\alpha}\ell^*(x)B^*(x) := \sup\{t : t^{-1/\alpha}\ell(t)B(t) > x\}, \qquad x > 0.$$

If $B(t)$, $t \in (0,1)$, is bounded away both from 0 and ∞, then so is $B^*(x)$, $x > 0$. If $B(t) \to 0$ as $t \downarrow 0$, then it is straightforward to see that $B^*(x) \to 0$ as $x \to \infty$. Since

$$F_-(x) = \sup\{t : Q(t) < x\}, \qquad x \in \mathbb{R},$$

we see that the first statement in (2.29) holds true if $M_1 \equiv 0$ and if $M_1 \not\equiv 0$ then $F_-(-x) = x^{-\alpha}\ell^*(x)B^*(x)$, $x > 0$, where, by (2.41), B^* is some function bounded away both from 0 and ∞. Our task therefore reduces to identifying B^*. It is well-known by classical theory (see [44], p. 116) that for the convergence in (2.8) it is necessary to have

$$k_nF(-A_{k_n}x) \Longrightarrow L(-x) = \frac{M_L(-x)}{x^{-\alpha}} \qquad \text{and} \qquad k_n(F(A_{k_n}x) - 1) \Longrightarrow R(x) = \frac{M_R(x)}{x^{-\alpha}},$$

$x > 0$.. This requirement is satisfied if $F_-(-x) = x^{-\alpha}\ell^*(x)[M_L(-\delta(x))]$, and so $h_L(x) := B^*(x) - M_L(-\delta(x))$ admits the property in (2.31), which is just the statement to be proved. For the second half of (2.29) a similar consideration applies and, finally, (2.32) can be seen in the same way as (2.25) in Theorem 2.3. ■

Proof of Theorem 2.4. (i) Suppose that (2.8) holds along some subsequence $\{k_n\}$ with a limiting random variable having a semistable law with exponent $\alpha \in (0,2]$. First we construct a subsequence $\{l_n\}$ along which (2.8) and (1.3a) hold simultaneously. Pick some $\{k'_n\} \subset \{k_n\}$ such that $\liminf_{n\to\infty} k'_{n+1}/k'_n > 1$ (if $\liminf_{n\to\infty} k_{n+1}/k_n > 1$ then $\{k'_n\}$ can be chosen as the whole original sequence $\{k_n\}$). Now, if $\mathbf{c} > 1$ and $\liminf_{n\to\infty} k'_{n+1}/k'_n < \infty$ then in the same way as in the proof of Theorem 2.2 we obtain that $\liminf_{n\to\infty} k'_{n+1}/k'_n = \mathbf{c}^{n_0}$ for some $n_0 \in \mathbb{N}$. Thus $\{k'_n\} = \{l_n\}$ is a satisfactory choice.

If $\liminf_{n\to\infty} k'_{n+1}/k'_n = \infty$ then we choose a $c > 1$, where $c = \mathbf{c}^{n_0}$ for some $n_0 \in \mathbb{N}$ if $\mathbf{c} > 1$, otherwise arbitrary. Set $l_{2n} := k'_n$ and $l_{2n+1} := \lfloor ck'_n \rfloor$, $A_{l_{2n+1}} := c^{1/\alpha}A_{k_n}$, $B_{l_{2n+1}} := cB_{k_n} + A_{l_{2n+1}}\left[\Theta(\psi_1^\alpha) - c^{-1/\alpha}\Theta({}^{(c)}\psi_1^\alpha) - \Theta(\psi_2^\alpha) + c^{-1/\alpha}\Theta({}^{(c)}\psi_2^\alpha)\right]$. The subsequence $\{l_n\}$ obviously satisfies (1.3a) with the given c and, by Lemma 2.3, (2.8) holds along $\{l_{2n+1}\}_{n=1}^\infty$, and hence along the whole $\{l_n\}_{n=1}^\infty$.

The statements in (2.33), (2.34) and (2.35) now follow from Theorem 6 (ii) in [19], using that in our case at least one of the ψ_j^α, $j = 1,2$ is not 0 on $[1,\infty)$.

(ii) The statement will be shown by a direct construction of an $F \in \mathbb{D}_{\mathrm{p}}(G)\setminus\mathbb{D}_{\mathrm{gp}}(G)$. In fact, we show somewhat more. We claim that for *any* given subsequence $\{k_n\}_{n=0}^\infty$, $k_{n+1}/k_n \to \infty$ there exists a quantile function $Q(\cdot)$ such that for the distribution function F determined by Q there exist normalizing and centring constants such that (2.8) holds along the given subsequence $\{k_n\}$, but $F \notin \mathbb{D}_{\mathrm{gp}}(G')$ for any distribution function G'.

We may assume without loss of generality that $k_0 > 2$ and $k_n > k_{n-1}$, $n \in \mathbb{N}$. Introduce

$$\theta_{1,n} := k_n^{-5/6} k_{n-1}^{-1/6} \quad \text{and} \quad \theta_{2,n} := k_n^{-5/6} k_{n+1}^{-1/6}$$

for $n \geq 1$. Obviously $\theta_{2,n} < 1/k_n < \theta_{1,n}$, $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{\theta_{1,n}}{1/k_n} = \lim_{n\to\infty} \frac{1/k_n}{\theta_{2,n}} = \infty. \tag{2.48}$$

We consider the case when both ψ_1^α and ψ_2^α are non-degenerate: the other one can be handled in a very similar manner. By this assumption there exist $\mathbf{C} < 1 < \mathbf{D}$ such that

$$0 < \mathbf{C} < \inf\{\min(M_1(s), M_2(s)) : s > 0\} < \sup\{\max(M_1(s), M_2(s)) : s > 0\} < \mathbf{D} < \infty.$$

Choose a threshold number n_0 such that

$$\exp(1/\theta_{2,n}) > \frac{\mathbf{D}}{\mathbf{C}} \exp(1/\theta_{1,n})(\theta_{1,n}/\theta_{2,n})^{1/\alpha}$$

holds all for $n \geq n_0$, and now define

$$Q(u) := \begin{cases} -\exp(1/u), & u \in (\theta_{1,n+1}, \theta_{2,n}],\ n \geq n_0, \\ -\exp(1/\theta_{1,n}) k_n^{1/\alpha} \left(\frac{k_n u}{\theta_{1,n}}\right)^{-\frac{1}{\alpha}} \frac{M_1^-(k_n u)}{\mathbf{C}}, & u \in (\theta_{2,n}, \theta_{1,n}],\ n \geq n_0, \\ 0, & u \in (\theta_{1,n_0}, 1-\theta_{1,n_0}], \\ \exp(1/\theta_{1,n}) k_n^{1/\alpha} \left(\frac{k_n(1-u)}{\theta_{1,n}}\right)^{-\frac{1}{\alpha}} \frac{M_2(k_n(1-u))}{\mathbf{C}}, & (1-u) \in [\theta_{2,n}, \theta_{1,n}),\ n \geq n_0, \\ \exp(1/(1-u)), & (1-u) \in [\theta_{1,n+1}, \theta_{2,n}),\ n \geq n_0, \end{cases}$$

where M_1^- is the left-continuous version of M_1. By the choice of n_0, Q is non-decreasing in $(0,1)$ and is left-continuous, i.e., Q is a proper quantile function. First we show that $F \notin \mathbb{D}_{\mathrm{p}}(2)$ for the distribution function F determined by Q.

Assume the contrary: then, by Corollary 1 (a) in [26],

$$\liminf_{s\downarrow 0} s^{1/2}\big(|Q_+(\lambda s)| + |Q(1-\lambda s)|\big)/\sigma(s) = 0$$

for all $0 < \lambda < \infty$. This in turn, by the representation in (2.4), implies that

$$\limsup_{s\downarrow 0} \frac{\int_s^{1/2} Q^2(u)du}{sQ^2(s)} = \infty \quad \text{or} \quad \limsup_{s\downarrow 0} \frac{\int_{1/2}^{1-s} Q^2(u)du}{sQ^2(1-s)} = \infty. \tag{2.49}$$

We show that $\int_s^{1/2} Q^2(u)du/\big[sQ^2(s)\big]$ is bounded as $s \downarrow 0$; the other side can be handled completely analogously. We distinguish two cases.

(1) Choose a threshold number n_1 such that $n_1 \geq n_0$ and $|Q(u)| < \frac{\mathbf{D}}{\mathbf{C}} \exp(\frac{1}{u})$ holds for $u \in (0, \theta_{1,n_1}]$. If $s \in (\theta_{1,n+1}, \theta_{2,n}]$ for some $n \geq n_1$ then by the definition of Q,

$$\begin{aligned}\frac{\int_s^{1/2} Q^2(u)du}{sQ^2(s)} &\leq \frac{\mathbf{D}^2}{\mathbf{C}^2} \frac{\int_s^{1/2} \exp(\frac{2}{u})du}{s\exp(\frac{2}{s})} + o(s) \\ &\leq \frac{2\mathbf{D}^2}{\mathbf{C}^2}\left[\frac{\exp(\frac{1}{s})\int_4^{1/s}\frac{1}{t^2}dt}{s\exp(\frac{2}{s})} + \frac{\exp(\frac{2}{s})\int_{1/s}^{2/s}\frac{1}{t^2}dt}{s\exp(\frac{2}{s})}\right] + o(s) \leq \frac{2\mathbf{D}^2}{\mathbf{C}^2}.\end{aligned} \tag{2.50}$$

The first term on the right-hand side of (2.50) converges to 0 as $s \downarrow 0$, and the second term is constantly $1/2$, so we obtain the bound $\frac{2\mathbf{D}^2}{\mathbf{C}^2}$ for all s sufficiently small.

(2) If $s \in (\theta_{2,n}, \theta_{1,n}]$ for some $n \geq n_1$ then, by the definition of Q, dropping the common constant factors in the numerator and denominator, we have

$$\frac{\int_s^{1/2} Q^2(u)du}{sQ^2(s)} \leq \frac{\mathbf{D}^2}{\mathbf{C}^2}\left[\frac{\int_s^{\theta_{1,n}} u^{-\frac{2}{\alpha}}du}{s^{1-\frac{2}{\alpha}}} + \frac{\int_{\theta_{1,n}}^{1/2}\exp(\frac{2}{u})du}{\theta_{1,n}\exp(\frac{2}{\theta_{1,n}})} + o(s)\right] \leq \left[\frac{1}{\frac{2}{\alpha}-1} + 2\right]\frac{\mathbf{D}^2}{\mathbf{C}^2}, \tag{2.51}$$

using (2.50) to estimate the second term. But (2.50) and (2.51), and the corresponding right-hand side variants together clearly contradict (2.49), therefore $F \notin \mathbb{D}_{\mathrm{p}}(2)$, hence $F \notin \mathbb{D}_{\mathrm{gp}}(2)$ holds true *a fortiori*.

Now we prove that $F \notin \mathbb{D}_{\mathrm{gp}}(G')$ for any G' nonnormal semistable law. Assume the contrary: Then by Theorem 2.3 there exists $\beta \in (0, 2)$ such that

$$K_1 \leq \liminf_{u\downarrow 0} \frac{Q(\lambda u)(\lambda u)^{1/\beta}}{Q(u)u^{1/\beta}} \leq \limsup_{u\downarrow 0} \frac{Q(\lambda u)(\lambda u)^{1/\beta}}{Q(u)u^{1/\beta}} \leq K_2 \tag{2.52}$$

for all $\lambda \in (0, \infty)$ fixed, where $0 < K_1 < K_2 < \infty$. However, (2.52) obviously fails for the Q defined above, since choosing for example $u_n := (k_n k_{n+1})^{-1/2}$ the lim inf on the left-hand side is 0 for $\lambda > 1$ and the lim sup on the right-hand side is infinite for $\lambda < 1$ along $\{u_n\}$. Thus, $F \notin \mathbb{D}_{\mathrm{gp}}(G')$ for any G' semistable.

On the other hand, $F \in \mathbb{D}_{\mathrm{p}}(G)$ along the given subsequence $\{k_n\}$, since by (2.48) for any $s \in (0, \infty)$ there exists an $N(s) \in \mathbb{N}$ such that $s/k_n \in (\theta_{2,n}, \theta_{1,n})$ for all $n \geq N(s)$. Then, setting $A_{k_n} := \mathbf{C}^{-1}(k_n\theta_{1,n})^{1/\alpha}\exp(1/\theta_{1,n})$,

$$\frac{Q_+\left(\frac{s}{k_n}\right)}{A_{k_n}} = -s^{1/\alpha}M_1(s) = \psi_1^\alpha(s) \quad \text{and} \quad \frac{-Q\left(1-\frac{s}{k_n}\right)}{A_{k_n}} = -s^{1/\alpha}M_2(s) = \psi_2^\alpha(s),$$

for all $n \geq N(s)$. The fact that $\psi_j^\alpha(s) \not\equiv 0$, $s \in [1, \infty)$, $j = 1, 2$, and $F \notin \mathbb{D}_{\mathrm{p}}(2)$ implies that the ratio $A_{k_n}/a(k_n)$ is bounded away both from 0 and ∞ as $n \to \infty$, where $a(k_n)$

is the natural normalizing sequence. Hence, by Theorem 1* of [19], for any subsequence $\{k_{n'}\} \subset \{k_n\}$ there exists a further subsequence $\{k_{n''}\} \subset \{k_{n'}\}$ such that we have

$$\frac{1}{A_{k_{n''}}}\left\{\sum_{j=1}^{k_{n''}} X_j - k_{n''}\mu_{0,0}(k_{n''})\right\} \xrightarrow{\mathcal{D}} V_{0,0}(\psi_1^\alpha, \psi_2^\alpha, \sigma''), \qquad \text{as} \quad n'' \to \infty, \tag{2.53}$$

with $\sigma'' < \infty$ generally depending on the chosen subsequence $\{k_{n''}\}$. But $\sigma'' \equiv 0$ since in the opposite case, by Corollary 9 of [26], $F \in \mathbb{D}_{\mathrm{p}}(2)$ would hold true, and we have already shown that $F \notin \mathbb{D}_{\mathrm{p}}(2)$. Thus (2.53) holds along the entire $\{k_n\}$. ■

2.4. The merge

By a special case of Theorem 2.3, if $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$, then

$$S(k_n) := \frac{\sum_{j=1}^{k_n} X_j - k_n \int_{\frac{1}{k_n}}^{1-\frac{1}{k_n}} Q(u)\,du}{k_n^{1/\alpha}\ell(1/k_n)} \xrightarrow{\mathcal{D}} V(\psi_1^\alpha, \psi_2^\alpha, 0). \tag{2.54}$$

One cannot in general replace k_n by n in (2.54), the resulting $\{S(n)\}_{n=1}^\infty$ typically does not converge in distribution and this is not a fault of the centralization or normalization, this is in the nature of things: it is the essence of semistability that we assumed convergence only along a subsequence and not the entire $\{n\}$. The main goal of this section is to show that the distribution functions of $S(n)$ nevertheless merge together, in fact uniformly, with the distribution functions of members of a family of semistable random variables around $V(\psi_1^\alpha, \psi_2^\alpha, 0)$. This is in fact obtained as a special case of a more general merge result in Theorem 2.6 for possibly lightly trimmed sums, following Theorem 2.5 of related interest on stochastic compactness; note that the effect of light trimming was first analyzed by Darling [37] and by Arov and Bobrov [4] when $F \in \mathbb{D}(\alpha)$ for some $\alpha \in (0,2)$. Analogous merging phenomena for sample extremes are in Theorem 2.7. Here we only study the behaviour of sample extremes when the (centralized and normalized) *sum* of the sample is assumed to converge (along $\{k_n\}$): i.e., $F \in \mathbb{D}_{\mathrm{gp}}(\mathcal{S}_*)$, where $\mathbb{D}_{\mathrm{gp}}(\mathcal{G}) = \bigcup_{G\in\mathcal{G}} \mathbb{D}_{\mathrm{gp}}(G)$ is the domain of geometric partial attraction of a class $\mathcal{G} \subset \mathcal{S}_*$. An essentially more general study of sample extremes will take place in Section 4.

Merging together, in the distributional sense, of properly centered and normalized sums and suitably chosen members of their subsequential limits was first discovered by Csörgő [21] for the cumulative gains in a series of St. Petersburg games; the merge theorem there is the basis for resolving the St. Petersburg paradox. Theorem 2.6 may serve as a basis to understand the general essence of the merging phenomenon.

For fixed choices of integers $l, m \geq 0$ the sequence of the lightly trimmed or whole (if $l = 0 = m$) sums $\sum_{l+1}^{n-m} X_{j,n}$, $n \in \mathbb{N}$, is said to be stochastically compact, when we write $F \in \mathbb{SC}(l,m)$, if there exists a sequence of normalizing constants $A_n > 0$ and a sequence of centering constants $C_n \in \mathbb{R}$ such that for every subsequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ there exists a further subsequence $\{n_{k_r}\}_{r=1}^{\infty} \subset \{n_k\}_{k=1}^{\infty}$ such that

$$\frac{1}{A_{n_{k_r}}}\left\{\sum_{j=l+1}^{n_{k_r}-m} X_{j,n_{k_r}} - C_{n_{k_r}}\right\} \xrightarrow{\mathcal{D}} W_{l,m} \quad \text{as} \quad r \to \infty, \tag{2.55}$$

for a non-degenerate random variable $W_{l,m}$, depending in general on the sequence $\{n_{k_r}\}$. Since $\mathbb{D}_{\text{gp}}(G_{0,0,\sigma}) = \mathbb{D}(2)$ for any $\sigma > 0$ and $\mathbb{D}(2)$ is well known by classical theory, we henceforth assume that $F \notin \mathbb{D}(2)$.

For $\psi \in \Psi$ and $\lambda > 0$ let ${}_\lambda\psi(s) = \psi(\lambda s)$, and put $\psi_j^{\alpha,\lambda}(s) = \lambda^{1/\alpha}\,{}_\lambda\psi_j^\alpha(s) = -M_j(\lambda s)s^{-1/\alpha}$, $s > 0$, for the $\psi_j^\alpha(\cdot)$ in (2.10), $j = 1,2$. Then, with the random variables taken from (2.2), define $W_{1,\alpha,\lambda}^{(l)}(M_1) = W_1^{(l)}(\psi_1^{\alpha,\lambda})$, $W_{2,\alpha,\lambda}^{(m)}(M_2) = W_2^{(m)}(\psi_2^{\alpha,\lambda})$ and finally $V_{\alpha,\lambda}^{l,m}(M_1,M_2) = V_{l,m}(\psi_1^{\alpha,\lambda},\psi_2^{\alpha,\lambda},0)$, and notice the identities $V_{\alpha,\lambda}^{l,m}(M_1,0) = W_{1,\alpha,\lambda}^{(l)}(M_1) = V_{\alpha,\lambda}^{l,0}(M_1,0)$, $V_{\alpha,\lambda}^{l,m}(0,M_2) = W_{2,\alpha,\lambda}^{(m)}(M_2) = V_{\alpha,\lambda}^{0,m}(0,M_2)$ and

$$V_{\alpha,\lambda}^{l,m}(M_1,M_2) = W_{2,\alpha,\lambda}^{(m)}(M_2) - W_{1,\alpha,\lambda}^{(l)}(M_1) = \lambda^{\frac{1}{\alpha}}\, V_{l,m}\big({}_\lambda\psi_1^\alpha,\, {}_\lambda\psi_2^\alpha, 0\big) \tag{2.56}$$

for all choices of the integers $l, m \geq 0$.

That $\mathbb{D}_{\text{gp}}(S_*) \subset \bigcap_{l,m\geq 0}\mathbb{SC}(l,m)$, which is the content of Theorem 2.5(i) below, can be seen by checking from (2.23) the necessary and sufficient condition (2.1.42a) of Corollary 10 in [26] for $F \in \mathbb{SC}$, from which corollary we also know that $\bigcap_{l,m\geq 0}\mathbb{SC}(l,m) = \mathbb{SC}(0,0)$. Theorem 2.5(ii), however, says much more: it is a necessary and sufficient condition for convergence in distribution along any given subsequence, from which part (i) comes in a trivial fashion. For this the full force of the assumption $F \in \mathbb{D}_{\text{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ is needed for some $\alpha \in (0,2)$ as described by Theorem 2.2 and Theorem 2.3. In particular, it is no loss of generality to assume that $F \in \mathbb{D}_{\text{gp}}^{(\mathbf{c})}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for the minimal common period $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) \geq 1$ from Theorem 2.2, that is, that (2.54) holds for $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ for which $k_{n+1}/k_n \to \mathbf{c}$, and it is then this sequence $\{k_n\}$ through which the function $\gamma(\cdot)$ in (2.23) is defined.

Theorem 2.5. *Suppose that $F \in \mathbb{D}_{\text{gp}}^{(\mathbf{c})}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for a semistable distribution $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of exponent $\alpha \in (0,2)$.*

(i) *Then $F \in \mathbb{SC}(l,m)$ for every pair of non-negative integers (l,m), and the norming and centering constants for (2.55) may be chosen as $A_n = n^{1/\alpha}\ell(1/n)$ and $C_n = n\int_{\frac{l+1}{n}}^{1-\frac{m+1}{n}} Q(s)\,ds$, $n \in \mathbb{N}$, where $\ell(\cdot)$ is from (2.23).*

(ii) *Suppose that for a subsequence $\{n_r\}_{r=1}^{\infty} \subset \mathbb{N}$ and norming and centering constants $A_{n_r} > 0$ and C_{n_r},*

$$\frac{1}{A_{n_r}}\left\{\sum_{j=1}^{n_r} X_{j,n_r} - C_{n_r}\right\} \xrightarrow{\mathcal{D}} W_{0,0}, \quad \text{as} \quad r \to \infty, \tag{2.57}$$

for a non-degenerate random variable $W_{0,0}$. *Then, necessarily, either the sequence* $\{\gamma(1/n_r)\}$ *converges to some* $\kappa \in [1,\mathbf{c}]$, *as* $r \to \infty$, *or has exactly two limit points,* 1 *and* $\mathbf{c}$ *(in which case we define* $\kappa = 1$*). With* $V^{0,0}_{\alpha,\kappa}(M_1, M_2)$ *as in (2.56), the limiting random variable necessarily satisfies* $W_{0,0} \stackrel{\mathcal{D}}{=} \delta\, V^{0,0}_{\alpha,\kappa}(M_1, M_2) + d$, *where*

$$\delta = \lim_{r\to\infty} \frac{n_r^{1/\alpha}\ell(1/n_r)}{A_{n_r}} > 0 \quad \text{and} \quad d = \lim_{r\to\infty} \frac{n_r \int_{\frac{1}{n_r}}^{1-\frac{1}{n_r}} Q(s)\, ds - C_{n_r}}{A_{n_r}}. \tag{2.58}$$

Conversely, if for a subsequence $\{n_r\}_{r=1}^{\infty} \subset \mathbb{N}$ *the sequence* $\{\gamma(1/n_r)\}$ *either converges to some* $\kappa \in [1,\mathbf{c}]$, *as* $r \to \infty$, *or has exactly two limit points,* 1 *and* $\mathbf{c}$ *(in which case we put* $\kappa = 1$*), then, with* $V^{l,m}_{\alpha,\kappa}(M_1, M_2)$ *as in (2.56),*

$$\frac{1}{n_r^{1/\alpha}\ell(1/n_r)} \left\{ \sum_{j=l+1}^{n_r - m} X_{j,n_r} - n_r \int_{\frac{l+1}{n_r}}^{1-\frac{m+1}{n_r}} Q(s)\, ds \right\} \xrightarrow{\mathcal{D}} V^{l,m}_{\alpha,\kappa}(M_1, M_2), \tag{2.59}$$

as $r \to \infty$, *for every pair* (l, m) *of non-negative integers.*

While true, the theorem is trivial for $\mathbf{c} = 1$ since $\mathbb{D}^{(1)}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) = \mathbb{D}(\alpha)$ and (2.59) in fact holds by Corollary 3 in [26] along the whole $\mathbb{N}$ for any integers $l, m \geq 0$. For $\mathbf{c} > 1$, (2.59) holds for all pairs (l, m) of non-negative integers along a given $\{n_r\}_{r=1}^{\infty} \subset \mathbb{N}$ if and only if $\lim_{r\to\infty} e^{[2\gamma(1/n_r)-\mathbf{c}-1]\pi i/[\mathbf{c}-1]} = e^{[2\kappa-\mathbf{c}-1]\pi i/[\mathbf{c}-1]}$ for some $\kappa \in [1, \mathbf{c}]$; convergence on the unit circle, the point $e^{-\pi i} = e^{\pi i}$ representing both 1 and $\mathbf{c}$.

An immediate consequence for the case $(l, m) = (0, 0)$, a version of which was first obtained by Mejzler [63], is that if $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for a semistable $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of index $\alpha \in (0, 2)$ given by (2.10), then the class of all non-degenerate subsequential limiting distributions of suitably centered and normalized full sums from F is contained in the family of distributions of the random variables $\{\delta\, V^{0,0}_{\alpha,\lambda}(M_1, M_2) + d \colon \lambda \geq 1, \delta > 0, d \in \mathbb{R}\}$. This implies that if $\mathbb{D}_{\mathrm{gp}}(G_1) \cap \mathbb{D}_{\mathrm{gp}}(G_2) \neq \emptyset$ for $G_1, G_2 \in \mathcal{S}_*$, then $\mathbb{D}_{\mathrm{gp}}(G_1) = \mathbb{D}_{\mathrm{gp}}(G_2)$.

The next corollary is also immediate. As an analogue of the classical convergence of types theorem ([44, §10]), it was recently proved by Meerschaert and Scheffler [59].

Corollary 2.5. *Suppose that (2.8) holds for a non-stable semistable limit and a subsequence* $\{k_n\}$ *for which* $\lim_{n\to\infty} k_{n+1}/k_n = c > 1$. *If (2.8) also holds, for a non-degenerate limit, along another subsequence* $\{k_n'\}$ *such that* $\lim_{n\to\infty} k_{n+1}'/k_n' = c$ *for the same* c, *then* $\lim_{n\to\infty} k_n'/k_n = \lambda$ *and* $\lim_{n\to\infty} A_{k_n'}/A_{k_n} = \delta$ *for some constants* $\lambda, \delta \in (0, \infty)$. ∎

In order to state the main result, introduce the distribution functions

$$F_n^{l,m}(x) = \mathbb{P}\left\{ \frac{1}{n^{1/\alpha}\ell(1/n)} \left[\sum_{j=l+1}^{n-m} X_{j,n} - n \int_{\frac{l+1}{n}}^{1-\frac{m+1}{n}} Q(s)\, ds \right] \leq x \right\}$$

and

$$G_{\kappa}^{l,m}(x) = \mathbb{P}\Big\{V_{\alpha,\kappa}^{l,m}(M_1, M_2) \le x\Big\}, \quad x \in \mathbb{R}, \tag{2.60}$$

for all $n \in \mathbb{N}$. With the same conventions as for Theorem 2.5, the merge results in the next theorem hold along the whole sequence of natural numbers.

Theorem 2.6. *If $F \in \mathbb{D}_{\mathrm{gp}}^{(\mathrm{c})}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for a semistable distribution $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of exponent $\alpha \in (0,2)$ and $\mathrm{c} = \mathrm{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) \ge 1$, then for every pair (l,m) of non-negative integers,*

$$\Delta_n^{l,m} = \sup_{x\in\mathbb{R}} \Big| F_n^{l,m}(x) - G_{\gamma(1/n)}^{l,m}(x) \Big| \to 0. \tag{2.61}$$

Remark. Using Theorem 3.3 from Section 3, it is also easily seen that for all pairs of sequences $\{q_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ of natural numbers such that $q_n \to \infty$, $r_n \to \infty$ and $q_n/n \to 0$, $r_n/n \to 0$,

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{S_n^{l,m}(r_n,q_n) - n\mu_n^{l,m}(r_n,q_n)}{n^{1/\alpha}\ell(1/n)} \le x \right\} - G_{\gamma(1/n)}^{l,m}(x) \right| \to 0,$$

where $S_n^{l,m}(r_n,q_n) = \sum_{j=l+1}^{r_n} X_{j,n} + \sum_{j=n-q_n+1}^{n-m} X_{j,n}$ and $\mu_n^{l,m}(r_n,q_n) = \int_{\frac{l+1}{n}}^{\frac{r_n}{n}} Q(s)\,ds + \int_{1-\frac{q_n}{n}}^{1-\frac{m+1}{n}} Q(s)\,ds$, and if $M_1(\cdot) \equiv 0$ and $M_2(\cdot) > 0$, then

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{\sum_{j=n-q_n+1}^{n-m} X_{j,n} - n\int_{1-\frac{q_n}{n}}^{1-\frac{m+1}{n}} Q(s)\,ds}{n^{1/\alpha}\ell(1/n)} \le x \right\} - G_{\gamma(1/n)}^{0,m}(x) \right| \to 0,$$

while if $M_1(\cdot) > 0$ and $M_2(\cdot) \equiv 0$, then

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \frac{\sum_{j=l+1}^{r_n} X_{j,n} - n\int_{\frac{l+1}{n}}^{\frac{r_n}{n}} Q(s)\,ds}{n^{1/\alpha}\ell(1/n)} \le x \right\} - G_{\gamma(1/n)}^{l,0}(x) \right| \to 0.$$

It is of course the case $(l,m) = (0,0)$ of (2.61) that gives merge with semistable laws for full sums. In general, for all possible choices of (l,m), the three extra statements show the fine structure in delineating those parts of the possibly lightly trimmed sums which govern the merging approximation: the rest with its share of the centering sequence is asymptotically negligible when normalized by what is needed for the whole sum. In the special case of $(l,m) = (0,0)$, these three statements together may be viewed as a generalization of Theorem 3 in [30], which is for $F \in \mathbb{D}(\alpha)$, $0 < \alpha < 2$.

The reason that merge takes place uniformly in Theorem 2.6 is that by Lemma 2.6 below $G_{\kappa}^{l,m}$ has a smooth density function for each $\kappa > 0$. Due to this fact, the distribution functions $F_{n_r}^{l,m}$ of the variables in (2.59) also converge uniformly to the limiting distribution function $G_\kappa^{l,m}$ there, as $r \to \infty$. In the trivial special case $\mathbf{c} = 1$ of Theorem 2.6, when $F \in \mathbb{D}_{\mathrm{gp}}^{(1)}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) = \mathbb{D}(\alpha)$, the respective sequence $G_{\gamma(1/n)}^{l,m}(x)$ can be replaced in all the statements by the limiting ("possibly trimmed stable") distribution function $G_1^{l,m}(x) = \mathbb{P}\{V_{l,m}(m_1\psi^\alpha, m_2\psi^\alpha, 0) \le x\}$, $x \in \mathbb{R}$, where the ingredients for the corresponding random variable $V_{l,m}(m_1\psi^\alpha, m_2\psi^\alpha, 0)$ in (2.2) are defined at the end of Section 2.1, and, in comparison with Corollary 3 in [26], the resulting uniformity of convergence is still new in the genuinely trimmed cases when $(l,m) \neq (0,0)$.

Observe also that $\cdots = G_{1/\mathbf{c}^2}^{l,m} = G_{1/\mathbf{c}}^{l,m} = G_1^{l,m} = G_{\mathbf{c}}^{l,m} = G_{\mathbf{c}^2}^{l,m} = \cdots$ for every pair (l,m) of non-negative integers, the source of which is the joint multiplicative periodicity of the two functions M_1 and M_2 in (2.10), the basic feature of semistability.

Turning now to extreme values, we shall use the characterization in Corollary 2.3 for an $F \in \mathbb{D}_{\mathrm{gp}}(\mathcal{S}_*)$ in terms of a sequence $\{k_n\}$ satisfying (1.3c). Introduce the distribution functions

$$H_n(x) = \mathbb{P}\left\{\frac{X_{n,n}}{n^{1/\alpha}\ell(1/n)} \le x\right\}, \quad x \in \mathbb{R},$$

for all $n \in \mathbb{N}$; and for $\kappa > 0$ and $M_R \not\equiv 0$ (which is equivalent to $\psi_2 \not\equiv 0$)

$$K_\kappa(x) = \begin{cases} 0, & x \le 0, \\ e^{-M_R(x/\kappa^{1/\alpha})/x^\alpha} = e^{R(x/\kappa^{1/\alpha})/\kappa}, & x > 0. \end{cases}$$

The merge theorem for maxima of variables from $\mathbb{D}_{\mathrm{gp}}(\mathcal{S}_*)$ is now as follows:

Theorem 2.7. *If $F \in \mathbb{D}_{\mathrm{gp}}^{(\mathbf{c})}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for a semistable distribution $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of exponent $\alpha \in (0,2)$, for which $\mathbf{c} = \mathbf{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0}) \ge 1$ and $M_2 \not\equiv 0$, then*

$$\mathcal{L}\big(H_n, K_{\gamma(1/n)}\big) \to 0. \tag{2.62}$$

Furthermore, if M_R is continuous, then (2.62) may be strengthened to

$$\sup_{x\in\mathbb{R}} \left|H_n(x) - K_{\gamma(1/n)}(x)\right| \to 0.$$

Here, as before, $\mathcal{L}(F,G)$ denotes the Lévy distance

$$\mathcal{L}(F,G) = \inf\{\varepsilon > 0\colon\ G(x-\varepsilon) - \varepsilon \le F(x) \le G(x+\varepsilon) + \varepsilon \text{ for all } x \in \mathbb{R}\}$$

of the distribution functions F and G.

We note that merging together of two sequences of probability measures is discussed in D'Aristotile, Diaconis, and Freedman [36] in very general terms. The abstract results there do not have implications for our problems here; the example in [36] nearest to what we do here is the investigation of the "wobbling maxima" of the integer parts of independent standard normal variables.

While the merging of cumulative St. Petersburg gains in [21] and the more general results in Theorem 2.6 above are the first appearances of (possibly lightly trimmed) sums of independent and identically distributed random variables merging together in distribution with a concretely defined sequence of approximate random variables, Arak's [2] famous theorem may also be viewed as a distributional merge theorem for sums of independent and identically distributed random variables. Greatly improving Kolmogorov's early constructions and some intermediate refinements, it states that the n^{th} convolution power F^{*n} of *any* distribution function on $\mathbb{R}$ can be uniformly approximated by a sequence of infinitely divisible distribution functions G_n at the rate of $n^{-2/3}$ and this merge rate is best in general; see also [3] for the proofs and references. However, Arak's deep theorem is practically an existence result: practically nothing more than its infinite divisibility is known about G_n. Naturally, one cannot expect that in the untrimmed case $(l,m)=(0,0)$ the merge in (2.61) would take place near to Arak's rate, i.e. that $G^{0,0}_{\gamma(1/n)}$ would determine the type of Arak's infinitely divisible approximation to F^{*n} for our special $F \in \mathbb{D}_{\text{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$, $\alpha \in (0,2)$. Nevertheless, it would still be of some interest to see whether the merge rates in (2.61), for $(l,m)=(0,0)$, achieve at least those rates of convergence to stable distributions that are known for the more restrictive cases when $F \in \mathbb{D}(\alpha)$, $\alpha \in (0,2)$.

Proofs

Although Fourier-analytic techniques also enter with Lemma 2.6 below, the proofs are still in the framework of the 'probabilistic' approach. Recall

$$\sigma^2(s\,,1-s)=\int_s^{1-s}\int_s^{1-s}\big[\min(u,v)-uv\big]\,dQ(u)\,dQ(v),\quad 0<s<\frac{1}{2},$$

from (2.4) and remember that if $F \in \mathbb{D}^{(c)}_{\text{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for a semistable $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of exponent $\alpha \in (0,2)$, then by the representation (2.23) in Theorem 2.3 and Lemma 2.4,

$$\limsup_{s\downarrow 0}\frac{\sigma(s\,,1-s)}{s^{\frac{1}{2}-\frac{1}{\alpha}}\ell(s)}<C \tag{2.63}$$

for some constant $C\in(0,\infty)$, where $\ell(\cdot)$ is the slowly varying function from (2.23), and that, recalling also for all $n\in\mathbb{N}$ and $0<s<n$ the functions

$$\psi_1(n\,,s)=\frac{Q_+\!\left(\frac{s}{n}\right)}{n^{1/\alpha}\ell(1/n)}\quad\text{and}\quad\psi_2(n\,,s)=\frac{-Q\!\left(1-\frac{s}{n}\right)}{n^{1/\alpha}\ell(1/n)}$$

from (2.7), the very reason why (2.54) holds along $\{k_n\}$ for which $k_{n+1}/k_n\to \mathrm{c}$ is that

$$\psi_j(k_n\,,s)\to\psi_j^\alpha(s),\quad j=1,2,\quad\text{and}$$

$$\lim_{u\to\infty}\limsup_{n\to\infty}\frac{\sqrt{k_n}\,\sigma\!\left(\frac{u}{k_n}\,,1-\frac{u}{k_n}\right)}{k_n^{1/\alpha}\ell(1/k_n)}=0 \tag{2.64}$$

at all the respective continuity points $s > 0$ of the limiting functions ψ_j^{α}, $j = 1, 2$, as we have shown in the sufficiency part of Theorem 2.3 above.

Lemma 2.5. *If $F \in \mathbb{D}_{\mathrm{gp}}^{(\mathrm{c})}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for a semistable $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of exponent $\alpha \in (0,2)$ and $M_{n,j}^*(\cdot) = M_j(\gamma(\cdot/k_n)) + h_j(\cdot/k_n)$ for the functions in (2.23), where $k_{n+1}/k_n \to \mathrm{c} = \mathrm{c}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$, and if $\{\tau_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that $\tau_n \to \tau$, where $\tau > 0$ is a continuity point of $M_j(\cdot)$, then $M_{n,j}^*(\tau_n) \to M_j(\tau)$, $j = 1, 2$.*

Proof. We only have to consider those cases of $j \in \{1, 2\}$ when $M_j \not\equiv 0$. Let $M_n^*(\cdot) = M(\gamma(\cdot/k_n)) + h(\cdot/k_n)$ be any one of $M_{n,j}^*(\cdot)$ for which this is the case, and let $\psi(k_n, \cdot)$ stand for the $\psi_j(k_n, \cdot)$ pertaining to this case. In this notation τ is a continuity point of $M(\cdot)$. We pick further continuity points $0 < s < \tau < t$, as close to τ as we wish. Then, by (2.64) and the fact that the quantile function Q is non-decreasing we see that $-s^{-1/\alpha} M(s) \le \liminf_{n\to\infty} \psi(k_n, \tau_n) \le \limsup_{n\to\infty} \psi(k_n, \tau_n) \le -t^{-1/\alpha} M(t)$. This implies that $\lim_{n\to\infty} \psi(k_n, \tau_n) = -\tau^{-1/\alpha} M(\tau)$. Since $\psi(k_n, \tau_n) = -\tau_n^{-1/\alpha} \ell(\tau_n/k_n) M_n^*(\tau_n)/\ell(1/k_n)$ for all n large enough, the slow variation of $\ell(\cdot)$ then implies $M_n^*(\tau_n) \to M(\tau)$, proving the lemma. ∎

Proof of Theorem 2.5. Since $1 \le \gamma(1/n) \le \mathrm{c} + \varepsilon$ for any $\varepsilon > 0$ if n is large enough, part (i) follows through the Bolzano – Weierstrass theorem from the sufficiency half of part (ii).

To prove that half first, consider any subsequence $\{n_r\}_{r=1}^{\infty} \subset \mathbb{N}$ for which the condition on $\{\gamma(1/n_r)\}$ is satisfied, with a κ as defined in the theorem. By Theorem 1* in [19], it suffices to show that the three relations in (2.64) hold for the sequence $\{n_r\}_{r=1}^{\infty}$ replacing $\{k_n\}$ and for the limiting functions $\psi_j^{\alpha,\kappa}(s) = -M_j(\kappa s)s^{-1/\alpha}$ for every $s > 0$ for which κs is a continuity point of $M_j(\cdot)$, $j = 1, 2$.

As to the third, by (2.63) and the slow variation of $\ell(\cdot)$ we see that

$$\limsup_{r\to\infty} \frac{\sqrt{n_r}\,\sigma\!\left(\frac{u}{n_r}, 1 - \frac{u}{n_r}\right)}{n_r^{1/\alpha}\ell(1/n_r)} \le \frac{C}{u^{\frac{2-\alpha}{2\alpha}}},$$

whence indeed

$$\lim_{u\to\infty} \limsup_{r\to\infty} \frac{\sqrt{n_r}\,\sigma\!\left(\frac{u}{n_r}, 1 - \frac{u}{n_r}\right)}{n_r^{1/\alpha}\ell(1/n_r)} = 0.$$

For the first two relations, noticing that $s/n_r = \gamma(1/n_r)s/k_{n^*(1/n_r)}$, by (2.23) we have

$$\psi_j(s, n_r) = -s^{-1/\alpha} \frac{\ell(s/n_r)}{\ell(1/n_r)} M^*_{n^*(1/n_r),j}(\gamma(1/n_r)s), \quad s > 0, \quad j = 1, 2,$$

in the notation of Lemma 2.5. Since $\lim_{r\to\infty} n^*(1/n_r) = \infty$, using the periodicity of M_j in the case of two limit points and the slow variation of $\ell(\cdot)$ again, Lemma 2.5 implies that $\lim_{r\to\infty} \psi_j(s, n_r) = -M_j(\kappa s)s^{-1/\alpha}$ whenever κs is a continuity point of $M_j(\cdot)$, $j = 1, 2$.

Conversely, suppose now (2.57) and choose any subsequence $\{n'_r\} \subset \{n_r\}$. Again by the Bolzano – Weierstrass theorem, there is a further subsequence $\{n''_r\} \subset \{n'_r\}$ such that $\gamma(1/n''_r) \to \kappa$ as $r \to \infty$ for some $\kappa \in [1, \mathbf{c}]$. But then by the sufficiency half we have (2.59) along $\{n''_r\}$ for $(l, m) = (0, 0)$, with the limiting random variable $V^{0,0}_{\alpha,\kappa}(M_1, M_2)$ for the given (α, M_1, M_2) determined by the condition that $F \in \mathbb{D}^{(\mathrm{c})}_{\mathrm{gp}}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$. We know from [26] and [19] that by the uniqueness property of Lévy's formula in (2.3) the representation of an infinitely divisible random variable in the first equation of (2.2) is also unique. This implies that the distributions of the variables $V^{0,0}_{\alpha,\kappa_1}(M_1, M_2)$ and $V^{0,0}_{\alpha,\kappa_2}(M_1, M_2)$ are of different type if $\kappa_1 \neq \kappa_2$, except for the case when $\kappa_1 = 1$ and $\kappa_2 = \mathbf{c}$, because then $V^{0,0}_{\alpha,1}(M_1, M_2) = V^{0,0}_{\alpha,\mathbf{c}}(M_1, M_2)$ even algebraically. Since we have an asymptotic distribution along the whole original $\{n_r\}$ in (2.57), it follows by the convergence of types theorem that either the subsequential limits of the sequence $\{\gamma(1/n_r)\}$ are the same, and then $\{\gamma(1/n_r)\}$ itself converges to some $\kappa \in [1, \mathbf{c}]$ as $r \to \infty$, or all the subsequential limits are either 1 or $\mathbf{c}$, in which case the sequence $\{\gamma(1/n_r)\}$ has exactly these two limit points. Thus we have both (2.57) and (2.59) along $\{n_r\}$, and so the two convergence relations in (2.58) also come from the convergence of types theorem. ■

Kruglov [55] proved that the distribution functions $G_{\psi_1^\alpha, \psi_2^\alpha, 0}$ of all semistable random variables $V(\psi_1^\alpha, \psi_2^\alpha, 0)$ are infinitely many times differentiable. Following from a special case of the following lemma, the first step towards Theorem 2.6 is an extension of this property to the distributional limits of all possibly lightly trimmed sums.

Lemma 2.6. *Let $\psi(s) = -M(s)s^{-1/\alpha}$, $0 < s < \infty$, where $\alpha \in (0, 2)$ and $M(\cdot)$ is any right-continuous function such that $C \leq M(s) \leq K$ for all $s > 0$ for some constants $0 < C \leq K < \infty$ and that $\psi(\cdot)$ is non-decreasing on $(0, \infty)$. Then for every non-negative integer r the common distribution function of the random variables $W_1^{(r)}(\psi)$ and $W_2^{(r)}(\psi)$ in (2.2) is infinitely many times differentiable on the whole line $\mathbb{R}$ and each of its derivatives has vanishing limits at $\pm\infty$.*

Proof. By Theorem 4 in [26], for the common characteristic function $\phi_{r,\psi}(t) = \mathbb{E}\big(\exp\{itW_j^{(r)}(\psi)\}\big)$, $j = 1, 2$, we have

$$\phi_{r,\psi}(t) = \int_0^\infty \exp\left\{\int_x^\infty \left[e^{it\psi(s)} - 1 - \frac{it\psi(s)}{1 + \psi^2(s)}\right] ds\right\} e^{it\rho_{r,\psi}(x)} \frac{x^r}{r!} e^{-x}\, dx, \qquad (2.65)$$

where

$$\rho_{r,\psi}(x) = \psi(x) + \int_{x+1}^{r+1} \psi(s)\, ds + \int_x^{x+1} \frac{\psi(s)}{1 + \psi^2(s)}\, ds - \int_{x+1}^\infty \frac{\psi^3(s)}{1 + \psi^2(s)}\, ds,$$

and hence

$$|\phi_{r,\psi}(t)| \leq \int_0^\infty \exp\left\{-\int_x^\infty \big[1 - \cos|t\psi(s)|\big] ds\right\} \frac{x^r}{r!}\, e^{-x} dx, \quad t \in \mathbb{R}.$$

If, for $x, t \geq 0$,

$$\frac{\pi x^{1/\alpha}}{2K} \leq t \quad \text{or, equivalently,} \quad x \leq \left(\frac{2Kt}{\pi}\right)^{\alpha},$$

then

$$\begin{aligned}
\int_x^{\infty} \left[1 - \cos|t\psi(s)|\right] ds &\geq \int_{(2Kt/\pi)^{\alpha}}^{\infty} \left[1 - \cos|t\psi(s)|\right] ds \\
&\geq \frac{4}{\pi^2}\, t^2 \int_{(2Kt/\pi)^{\alpha}}^{\infty} \psi^2(s)\, ds \\
&\geq \frac{4C^2}{\pi^2}\, t^2 \int_{(2Kt/\pi)^{\alpha}}^{\infty} s^{-\frac{2}{\alpha}}\, ds \\
&= \frac{4C^2}{\pi^2}\, \frac{\alpha}{2-\alpha} \left(\frac{\pi}{2K}\right)^{2-\alpha} t^{\alpha} =: K_{\alpha}\, t^{\alpha}
\end{aligned}$$

because if $s \geq (2Kt/\pi)^{\alpha}$, then

$$0 \leq \frac{|t|C}{s^{1/\alpha}} \leq \left|t\psi(s)\right| \leq \frac{|t|K}{s^{1/\alpha}} \leq \frac{\pi}{2},$$

and, following for example from the well-known product expansion

$$\cos y = \prod_{k=1}^{\infty} \left(1 - \frac{4y^2}{(2k-1)^2\pi^2}\right), \quad y \in \mathbb{R},$$

we have $1 - \cos y \geq 4y^2/\pi^2$ if $0 \leq y \leq \pi/2$. Therefore,

$$\begin{aligned}
\int_0^{\infty} t^k \left|\phi_{r,\psi}(t)\right| dt &\leq \int_0^{\infty} \left[\int_0^{\infty} t^k \exp\left\{-\int_x^{\infty} \left[1 - \cos|t\psi(s)|\right] ds\right\} dt\right] \frac{x^r}{r!}\, e^{-x} dx \\
&\leq \int_0^{\infty} \left[\int_0^{\frac{\pi x^{1/\alpha}}{2K}} t^k\, dt + \int_{\frac{\pi x^{1/\alpha}}{2K}}^{\infty} t^k\, e^{-K_{\alpha} t^{\alpha}}\, dt\right] \frac{x^r}{r!}\, e^{-x} dx \\
&< \infty
\end{aligned}$$

for any non-negative integer k, which by a standard Fourier-analytic result (see in [16, p. 438], for example) implies that the distribution function pertaining to $\phi_{r,\psi}(\cdot)$ is $k+1$ times differentiable with derivatives vanishing at $\pm\infty$. Since $k \geq 0$ is arbitrary, the lemma follows. ■

Proof of Theorem 2.6. Notice first that for any fixed integers $l, m \geq 0$ the parametric family $\{G_{\kappa}^{l,m} : \kappa > 0\}$ of distribution functions defined in (2.60) is continuous with respect to weak convergence: $\mathcal{L}(G_{\kappa_n}^{l,m}, G_{\kappa}^{l,m}) \to 0$ for any $\kappa > 0$ whenever $\kappa_n \to \kappa$. This can be seen in two different ways. One is, with reference to the continuity theorem of characteristic functions, to check that for each fixed $t \in \mathbb{R}$ the characteristic function $\int_{\mathbb{R}} e^{itx} dG_{\kappa}^{l,m}(x) = \mathbb{E}\big(\exp\{itV_{\alpha,\kappa}^{l,m}(M_1, M_2)\}\big) = \phi_{l,\psi_1^{\alpha,\kappa}}(-t)\phi_{m,\psi_2^{\alpha,\kappa}}(t)$ is continuous in the parameter $\kappa > 0$. Substituting $\psi_j^{\alpha,\kappa}(s) = -s^{-1/\alpha} M_j(\kappa s)$, $s > 0$, $j = 1, 2$, given at (2.56), into the formula in (2.65), this can be seen readily for both factors. The other way is to check that the stochastic process $\{V_{\alpha,\kappa}^{l,m}(M_1, M_2) : \kappa > 0\}$ does not have fixed discontinuities, that is, $V_{\alpha,\kappa_n}^{l,m}(M_1, M_2) \to V_{\alpha,\kappa}^{l,m}(M_1, M_2)$ almost surely whenever $\kappa_n \to \kappa > 0$, which is elementary using (2.56) and the representation in (2.2).

Now, to prove (2.61), pick any subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$. Then by the Bolzano – Weierstrass theorem this contains a further subsequence $\{n_{j_r}\}_{r=1}^{\infty}$ such that $\lim_{r\to\infty} \gamma(1/n_{j_r}) = \kappa$ for some $\kappa \in [1, \mathbf{c}]$. Then by Theorem 2.5 and by what we just proved,

$$\mathcal{L}\Big(F_{n_{j_r}}^{l,m}, G_{\kappa}^{l,m}\Big) \to 0 \quad \text{and} \quad \mathcal{L}\Big(G_{\gamma(1/n_{j_r})}^{l,m}, G_{\kappa}^{l,m}\Big) \to 0 \quad \text{as} \quad r \to \infty.$$

By Theorem 2.3 and by an application of Lemma 2.6 we see that the distribution function $G_{\kappa}^{l,m}$ is uniformly continuous for every $\kappa \in [1, \mathbf{c}]$, in fact the family $\{G_{\kappa}^{l,m} : \kappa > 0\}$ is uniformly equicontinuous as can be seen by a simple variant of Lemma 2.6, and hence the last two relations can be strengthened to

$$\sup_{x\in\mathbb{R}} \Big|F_{n_{j_r}}^{l,m}(x) - G_{\kappa}^{l,m}(x)\Big| \to 0 \quad \text{and} \quad \sup_{x\in\mathbb{R}} \Big|G_{\gamma(1/n_{j_r})}^{l,m}(x) - G_{\kappa}^{l,m}(x)\Big| \to 0,$$

as $r \to \infty$, which by the triangle inequality for the uniform distance implies that $\lim_{r\to\infty} \Delta_{n_{j_r}}^{l,m} = 0$. Since the subsequence $\{n_j\} \subset \mathbb{N}$ was arbitrary, we have $\lim_{n\to\infty} \Delta_n^{l,m} = 0$, proving (2.61). ∎

Proof of Theorem 2.7. Since $H_n(0) \to 0 = K_{\gamma(1/n)}(0)$, to prove (2.62) it clearly suffices to show that for every $\varepsilon > 0$,

$$H_n(x - \varepsilon) - \varepsilon \le K_{\gamma(1/n)}(x) \le H_n(x + \varepsilon) + \varepsilon \quad \text{for all } x > 0, \tag{2.66}$$

for all n large enough.

Considering any $x > 0$ and setting $A_n = n^{1/\alpha}\ell(1/n)$, by (2.29) we have

$$\begin{aligned} H_n(x) &= F^n(A_n x) \\ &= \left[1 - \frac{x^{-\alpha}\big(M_R(\delta(A_n x)) + h_R(A_n x)\big)}{n} \, \frac{\ell^*(x n^{1/\alpha}\ell(1/n))}{\ell(1/n)^{\alpha}}\right]^n, \end{aligned}$$

where $\delta(\cdot)$, and $\gamma(\cdot)$ above, are defined in terms of the basic $\{k_n\}$ sequence for which $k_{n+1}/k_n \to \mathbf{c}$. The inversion formula in (2.30) implies that $\ell^*(n^{1/\alpha}\ell(1/n))/\ell^{\alpha}(1/n) \to 1$, whence $\ell^*(x n^{1/\alpha}\ell(1/n))/\ell^{\alpha}(1/n) \to 1$. Now, pick any unbounded subsequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$. Then, since the sequence $\{\gamma(1/n_j)\}$ is bounded with all of its limit points in the interval $[1, \mathbf{c}]$, by the Bolzano – Weierstrass theorem there exists a further subsequence $\{n_{j_r}\}_{r=1}^{\infty}$ such that $\{\gamma(1/n_{j_r})\}$ converges to some $\kappa \in [1, \mathbf{c}]$, as

$r \to \infty$. Then we have the asymptotic equality $M_R(\delta(A_{n_{j_r}}x)) = M_R(A_{n_{j_r}}x/a(A_{n_{j_r}}x)) \sim M_R(x/\gamma^{1/\alpha}(1/n_{j_r}))$ as $r \to \infty$, provided that $x/\kappa^{1/\alpha}$ is a continuity point of M_R, in the proof of which we use the equation $M_R(\mathbf{c}^{1/\alpha}t) = M_R(t)$ for various choices of $t > 0$. Indeed, by a suitable analogue of Lemma 2.5, presently based on the classical fact that $k_n[1 - F(A_{k_n}t)] \to M_R(t)$ at every continuity point $t > 0$ of M_R, we even see that the error term $h_R(A_{n_{j_r}}x)$ is negligible, so that $M_R(\delta(A_{n_{j_r}}x)) + h_R(A_{n_{j_r}}x) \to M_R(x/\kappa^{1/\alpha})$ as $r \to \infty$. Whence

$$H_{n_{j_r}}(x) \to K_\kappa(x) = e^{-M_R(x/\kappa^{1/\alpha})/x^\alpha} \quad \text{as } r \to \infty \tag{2.67}$$

at every $x \in \mathbb{R}$ for which $x/\kappa^{1/\alpha}$ is a continuity point of M_R.

We infer from this property that (2.66) holds. Suppose the contrary. Then there exist a bad $\varepsilon > 0$, a bad $x_\varepsilon > 0$ and an unbounded sequence $\{n_j\}_{j=1}^\infty \subset \mathbb{N}$ such that either

$$\begin{gathered} H_{n_j}(x_\varepsilon - \varepsilon) - \varepsilon > K_{\gamma(1/n_j)}(x_\varepsilon) \quad \text{for all } j \in \mathbb{N}, \\ \text{or} \\ H_{n_j}(x_\varepsilon + \varepsilon) + \varepsilon < K_{\gamma(1/n_j)}(x_\varepsilon) \quad \text{for all } j \in \mathbb{N}, \end{gathered} \tag{2.68}$$

or both. Let $\{n_{j_r}\}$ be a subsequence for which $\lim_{r\to\infty} \gamma(1/n_{j_r}) = \kappa \in [1, \mathbf{c}]$, so that (2.67) holds. If the first case in (2.68) is not empty, choose any $x_- \in (x_\varepsilon - \varepsilon, x_\varepsilon)$ such that $x_-/\kappa^{1/\alpha}$ is a continuity point of M_R. Then

$$\begin{aligned} \limsup_{r\to\infty} H_{n_{j_r}}(x_\varepsilon - \varepsilon) - \varepsilon &\le \lim_{r\to\infty} H_{n_{j_r}}(x_-) - \varepsilon \\ &= K_\kappa(x_-) - \varepsilon < \liminf_{r\to\infty} K_{\gamma(1/n_{j_r})}(x_\varepsilon), \end{aligned}$$

which contradicts (2.68). Choosing any $x_+ \in (x_\varepsilon, x_\varepsilon + \varepsilon)$ for a non-empty second case in (2.68), the contradiction arises in the same way. Thus (2.66) and hence (2.62) follow. Using the uniform version of (2.67), the proof of the second statement for the sup-distance is analogous. ■

3. Moderately trimmed sums

It was shown in the previous section that the existence and the nature of asymptotic distributions of the lightly trimmed sums $S_{n_k}(l,m) = \sum_{j=l+1}^{n_k-m} X_{j,n_k}$, for fixed pairs of positive integers l and m is closely connected with those of the limiting distributions of the whole untrimmed sums $S_{n_k} = S_{n_k}(0,0) = \sum_{j=1}^{n_k} X_j$. We shall focus now our attention on the *moderately trimmed* sums $S_n(l_n, m_n) = \sum_{j=l_n+1}^{n-m_n} X_{j,n}$, where

$$l_n \to \infty, \ \frac{l_n}{n} \to 0 \quad \text{and} \quad m_n \to \infty, \ \frac{m_n}{n} \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Concerning moderately trimmed sums, the first deeper result is due to Csörgő, Horváth and Mason [30], who proved that if the full sums S_n have a nondegenerate asymptotic distribution along the whole $\{n\} = \mathbb{N}$, i.e. if F is in the domain of attraction of a (normal or nonnormal) stable law, then with $l_n \equiv m_n$ and suitable centering and norming sequences $S_n(m_n, m_n)$ is asymptotically normal as $n \to \infty$. Csörgő, Haeusler, and Mason [27] then determined the class of all possible asymptotic distributions for $S_n(l_n, m_n)$ along all possible subsequences $\{n_k\}$, together with necessary and sufficient conditions for the convergence in distribution of $S_{n_k}(l_{n_k}, m_{n_k})$ as $k \to \infty$. To formulate at least the condition for asymptotic normality, for $0 < s < 1-t < 1$ recall from (2.4) the function

$$\begin{aligned}
\sigma^2(s\,, 1-t) &= \int_s^{1-t}\int_s^{1-t} \left[\min(u,v) - uv\right] dQ(u)\, dQ(v) \\
&= sQ^2(s) + tQ^2(1-t) + \int_s^{1-t} Q^2(u)\, du \\
&\quad - \left[sQ(s) + tQ(1-t) + \int_s^{1-t} Q(u)\, du\right]^2,
\end{aligned}$$

and for given sequences $\{l_n\}$ and $\{m_n\}$ set

$$a_n(l_n, m_n) = \sqrt{n}\,\sigma\!\left(\frac{l_n}{n}\,, 1 - \frac{m_n}{n}\right), \tag{3.2}$$

introduce the two sequences of functions

$$\varphi_{1,n}(x) = \begin{cases} \varphi_{1,n}\!\left(-\frac{\sqrt{l_n}}{2}\right), & -\infty < x < -\frac{\sqrt{l_n}}{2}, \\ \frac{\sqrt{l_n}}{a_n(l_n,m_n)}\left\{Q\!\left(\frac{l_n}{n} + x\frac{\sqrt{l_n}}{n}\right) - Q\!\left(\frac{l_n}{n}\right)\right\}, & -\frac{\sqrt{l_n}}{2} \le x \le \frac{\sqrt{l_n}}{2}, \\ \varphi_{1,n}\!\left(\frac{\sqrt{l_n}}{2}\right), & \frac{\sqrt{l_n}}{2} < x < \infty, \end{cases}$$

and

$$\varphi_{2,n}(x) = \begin{cases} \varphi_{2,n}\Big(-\frac{\sqrt{m_n}}{2}\Big), & -\infty < x < -\frac{\sqrt{m_n}}{2}, \\ \frac{\sqrt{m_n}}{a_n(l_n,m_n)}\Big\{Q\Big(1-\frac{m_n}{n}+x\frac{\sqrt{m_n}}{n}\Big) - Q\big(1-\frac{m_n}{n}\big)\Big\}, & -\frac{\sqrt{m_n}}{2} \le x \le \frac{\sqrt{m_n}}{2}, \\ \varphi_{2,n}\Big(\frac{\sqrt{m_n}}{2}\Big), & \frac{\sqrt{m_n}}{2} < x < \infty, \end{cases}$$

and let Z be a standard normal random variable. Then, according to Theorem 4 of [27], there exist for the sequences $\{l_n\}$ and $\{m_n\}$, satisfying (3.1), centering and normalizing constants $C_n \in \mathbb{R}$ and $A_n > 0$ such that $A_n^{-1}[S_n(l_n,m_n) - C_n] \xrightarrow{\mathcal{D}} Z$ as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \varphi_{j,n}(x) = 0 \quad \text{for every } x \in \mathbb{R},\ j = 1,2, \tag{3.3}$$

in which case $C_n \equiv c_n(l_n,m_n) := n \int_{\frac{l_n+1}{n}}^{1-\frac{m_n+1}{n}} Q(u)\,du$ and $A_n \equiv a_n(l_n,m_n)$ work.

The subsequential version of this result is also true. If at least one of the functions $\varphi_{j,n}(\cdot)$, or one of the renormalized functions $a_n(l_n,m_n)\varphi_{j,n}(\cdot)/A_n$ for some $A_n > 0$ for which $a_n(l_n,m_n)/A_n \to 0$, $j = 1,2$, converges to a nonzero function either along the whole $\{n\}$ or along a subsequence, then extra terms appear in the limiting random variable so that the asymptotic distribution, typically obtained along a further subsequence, is no longer normal; it does not even have a normal component in the renormalized case. The conditions appearing are optimal; for the precise statements the reader is referred to [27] and [29]. Griffin and Pruitt [45] rederived this theory by a different method, obtaining the conditions and the description of limiting random variables in alternative forms, with numerous additional observations.

While the "asymptotic continuity" condition (3.3) solves the problem of asymptotic normality of moderately trimmed sums completely from a general mathematical point of view, its probabilistic meaning is not so clear until it is tied to better understood conditions that govern the asymptotic distribution of the entire untrimmed sums. Indeed, it was pointed out in [27] and then in [45] that if F is stochastically compact, meaning that the full sums are stochastically compact in the sense that there exist sequences of constants $b_n \in \mathbb{R}$ and $d_n > 0$ such that every subsequence of $\mathbb{N}$ contains a further subsequence along which $[S_n - b_n]/d_n$ converges in distribution to a nondegenerate random variable, then the sequences of functions $\{\varphi_{j,n}(\cdot)\}_{n=1}^{\infty}$ are uniformly bounded, $j = 1,2$, and hence the sequence $S_n^*(l_n,m_n) := [S_n(l_n,m_n) - c_n(l_n,m_n)]/a_n(l_n,m_n)$ of centered and normed trimmed sums is also stochastically compact for any pair (l_n,m_n) of sequences satisfying (3.1). However, nonnormal subsequential limiting distributions do arise in this case.

The only explicitly determined family of underlying distributions for which $S_n^*(m_n,m_n)$ is known to be asymptotically normal along the whole $\mathbb{N}$ for *every* sequence $\{m_n\}$ satisfying (3.1) is the family of those F that are in the domain of attraction of an arbitrary stable law (Csörgő, Horváth and Mason [30]), and the only explicit family for which $S_n^*(l_n, m_n)$ is known to be asymptotically normal along the whole $\mathbb{N}$ for *every* sequence $\{(l_n, m_n)\}$ of pairs satisfying (3.1) is the subfamily attracted by not completely asymmetric stable laws (Griffin and Pruitt [45]). The question arises whether there is a probabilistically meaningful larger class of distributions, necessarily within the class of stochastically compact distributions, which would respectively contain the families above and for which the same conclusions for the asymptotic normality of trimmed sums would still hold true. A feature of the phenomenon would of course be that the full sums, $[S_n - b_n]/d_n$, would no longer converge in distribution themselves along the whole $\{n\} = \mathbb{N}$. The aim of this section is to show that a larger class of distributions within the class of stochastically compact distributions does indeed exist with these properties: it is a proper subfamily of the family of distributions in the domain of geometric partial attraction of semistable laws. Recalling the all the notation from the previous section, our main result is as follows.

Theorem 3.1. *Suppose that $F \in \mathbb{D}_{\mathrm{gp}}(G)$ for some nondegenerate semistable law $G = G_{\psi_1,\psi_2,\sigma}$ such that both ψ_1 and ψ_2 are continuous on $(0,\infty)$.*

(i) *If neither of ψ_1 and ψ_2 is identically zero, then for any two sequences $\{l_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$ of positive integers satisfying (3.1),*

$$\frac{1}{a_n(l_n, m_n)} \left\{ \sum_{j=l_n+1}^{n-m_n} X_{j,n} - n \int_{\frac{l_n}{n}}^{1-\frac{m_n}{n}} Q(u)\, du \right\} \xrightarrow{\mathcal{D}} Z, \tag{3.4}$$

where $a_n(l_n, m_n)$ is as in (3.2) and Z is a standard normal random variable.

(ii) *If at least one of ψ_1 and ψ_2 is identically zero, then (3.4) holds true for any two sequences $\{l_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$ of positive integers satisfying (3.1) such that*

$$0 < \liminf_{n\to\infty} \frac{l_n}{m_n} \le \limsup_{n\to\infty} \frac{l_n}{m_n} < \infty. \tag{3.5}$$

The distribution G in the theorem is either normal, i.e. $G = G_{0,0,\sigma}$ for some $\sigma > 0$, in which case the continuity condition is trivially satisfied and part (ii) for this case is just a restatement of part of Theorem 1 in [30] when $l_n \equiv m_n$, or $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$, a semistable law with exponent $\alpha \in (0,2)$ with continuous ψ_1^α and ψ_2^α satisfying (2.10), in which case the two cases extend results in [27] and [45] mentioned above. Through (2.10), the condition is the continuity of the corresponding functions M_1 and M_2. This condition cannot be dropped in general as the example of the St. Petersburg game shows, where the underlying distribution is in the domain of geometric partial attraction of a semistable law with exponent 1 and Theorem 3.2 of Csörgő and Dodunekova [23] shows that nonnormal limits do arise for moderately trimmed sums along subsequences of $\mathbb{N}$. The generalized St. Petersburg games considered by Csörgő and Simons [34] in a different context and their symmetrized versions may serve to show the same for all exponents $\alpha \in (0,2)$. In terms of the Lévy functions L and R in (2.3), we see that a nonzero ψ_1^α (or ψ_2^α) is continuous, or equivalently the corresponding M_1 (or M_2) is continuous if and only if L (or R) does not have flat stretches in the sense that it is not constant on intervals with positive length.

We emphasize that even though (2.26) holds for the full sums only along a subsequence satisfying (1.3c), the convergence in (3.4) takes place along the whole $\mathbb{N}$. Thus, similarly as in Theorem 2.6 (the Merge Theorem), a condition along a subsequence was enough to yield a statement valid along the whole $\mathbb{N}$. If the continuity condition is violated, we still have an existence result along the whole $\mathbb{N}$.

Theorem 3.2. *If $F \in \mathbb{D}_{\mathrm{gp}}(G)$ for a nondegenerate semistable law G, then there exist two sequences $\{l_n\}_{n=1}^\infty$ and $\{m_n\}_{n=1}^\infty$ of integers satisfying (3.1) such that (3.4) holds.*

With some extra work the proof can be modified to allow the choice $l_n \equiv m_n$. Also, if $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ for some exponent $\alpha \in (0,2)$, neither of ψ_1^α and ψ_2^α is identically zero and ψ_1^α is continuous, then there is an $\{m_n\}_{n=1}^\infty$ satisfying (3.1) such that (3.4) holds for every $\{l_n\}_{n=1}^\infty$ satisfying (3.1); an analogous statement is true when ψ_2^α is continuous.

Our last result extends Theorem 3 in [30] and demonstrates that asymptotic semistability in (2.26) is determined only by arbitrarily small moderate portions of upper and lower order statistics in the sample.

Theorem 3.3. *If $F \in \mathbb{D}_{\mathrm{gp}}(G)$ for a nonnormal semistable law $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of exponent $\alpha \in (0,2)$, so that (2.26) holds along a subsequence $\{k_n\}_{n=1}^\infty \subset \mathbb{N}$ satisfying (1.3c), then*

$$\frac{1}{k_n^{1/\alpha}\ell(1/k_n)}\left\{\sum_{j=1}^{l_{k_n}} X_{j,k_n} - k_n\int_{\frac{1}{k_n}}^{\frac{l_{k_n}}{n}} Q(u)\,du\right\} \xrightarrow{\mathcal{D}} -W_1(\psi_1^\alpha), \tag{3.6}$$

$$\frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=l_n+1}^{n-m_n} X_{j,n} - n\int_{\frac{l_n}{n}}^{1-\frac{m_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0, \tag{3.7}$$

where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability, and

$$\frac{1}{k_n^{1/\alpha}\ell(1/k_n)}\left\{\sum_{j=k_n-m_{k_n}+1}^{k_n} X_{j,k_n} - k_n\int_{1-\frac{m_{k_n}}{k_n}}^{1-\frac{k_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathcal{D}} W_2(\psi_2^\alpha), \tag{3.8}$$

where the independent random variables $W_1(\psi_1^\alpha)$ and $W_2(\psi_1^\alpha)$ are given at (2.2), and so

$$\frac{1}{k_n^{1/\alpha}\ell(1/k_n)}\left\{\sum_{j=1}^{l_{k_n}} X_{j,k_n} + \sum_{j=k_n-m_{k_n}+1}^{k_n} X_{j,k_n} - k_n\left[\int_{\frac{1}{k_n}}^{\frac{l_{k_n}}{n}} Q(u)\,du + \int_{1-\frac{m_{k_n}}{k_n}}^{1-\frac{k_n}{n}} Q(u)\,du\right]\right\}$$

$$\xrightarrow{\mathcal{D}} V(\psi_1^\alpha,\psi_2^\alpha,0) = -W_1(\psi_1^\alpha) + W_2(\psi_2^\alpha)$$

for any two sequences $\{l_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$ of positive integers satisfying (3.1).

The general theory in [26], [29] and [19] ensures the *existence* of sequences $\{l_n\}$ and $\{m_n\}$ satisfying (3.1) for which these statements hold, the point of Theorem 3.3 is that they hold for *all* such sequences. If $M_j = \psi_j^\alpha \equiv 0$, which is allowed in (2.26) and in Theorem 3.3 above for one of the j, then of course $W_j(0) = 0$. A more general version of Theorem 3.3, in which a fixed number of the smallest and the largest extremes may be discarded from the sums in (3.6) and (3.8) is also true; the way in which the centering sequences and the limiting random variables should be changed in (3.6) and (3.8) for this version is clear from Section 2. This is the version of Theorem 3.3 that generalizes Corollary 2.1 in every respect. The formulation of theorem above suits well the genuinely two-sided case. In the completely asymmetric case when one of ψ_1^α and ψ_2^α is identically zero, a somewhat stronger statement can be made, even in the more general version with possible light trimming: see the end of the proof of Theorem 3.3 for this in the present case of full extreme sums.

Proofs

The proofs of the theorems will greatly rely on Lemma 2.4 from Section 2: observe that by the consideration in the proof of the sufficiency part of Theorem 2.3, if $F \in \mathbb{D}_{\text{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ the quantile function $Q(\cdot)$ pertaining to $F(\cdot)$ satisfies (2.41).

Proof of Theorem 3.1. To prove part (i), consider any two sequences $\{l_n\}_{n=1}^{\infty}$ and $\{m_n\}_{n=1}^{\infty}$ satisfying (3.1) and introduce the "renormalized" half-sided functions

$$\varphi_{n,l_n}^{(1)}(x) = \frac{a_n(l_n,m_n)}{a_{1,n}(l_n)}\,\varphi_{1,n}(x) \quad \text{and} \quad \varphi_{n,m_n}^{(2)}(x) = \frac{a_n(l_n,m_n)}{a_{2,n}(m_n)}\,\varphi_{2,n}(x), \quad x \in \mathbb{R},$$

the original functions $\varphi_{1,n}(\cdot)$ and $\varphi_{2,n}(\cdot)$ being given between (3.2) and (3.3), where

$$a_{1,n}(l_n) = \sqrt{n}\,\sigma\Big(\frac{l_n}{n},\frac{1}{2}\Big) \quad \text{and} \quad a_{2,n}(m_n) = \sqrt{n}\,\sigma\Big(\frac{1}{2},1-\frac{m_n}{n}\Big).$$

Since none of $\psi_1 := \psi_1^\alpha$ and $\psi_2 := \psi_2^\alpha$ is zero anywhere, $0<\alpha<2$, it follows from (2.23) and (2.4) that $a_{1,n}(l_n), a_{2,n}(m_n) > 0$, and so the definitions of the renormalized functions are meaningful for all n large enough and, of course, $a_{1,n}(l_n), a_{2,n}(m_n) \le a_n(l_n,m_n)$. Hence, to prove (3.3), it suffices to show that

$$\lim_{n\to\infty} \varphi_{n,l_n}^{(1)}(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi_{n,m_n}^{(2)}(x) = 0 \quad \text{for every } x \in \mathbb{R}. \tag{3.9}$$

To deal with $\varphi_{n,l_n}^{(1)}(x)$ at any fixed $x \in \mathbb{R}$, note that by the domain theorem at (2.23),

$$Q\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big) = -\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)^{-\frac{1}{\alpha}}\ell\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big[M_1\Big(\gamma\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big)+h_1\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big]$$

and

$$Q\Big(\frac{l_n}{n}\Big) = -\Big(\frac{l_n}{n}\Big)^{-\frac{1}{\alpha}} \ell\Big(\frac{l_n}{n}\Big)\Big[M_1\Big(\gamma\Big(\frac{l_n}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n}\Big)\Big]$$

for all n large enough. We substitute these into the formula for $\varphi^{(1)}_{n,l_n}(x)$ through the formula given for $\varphi_{1,n}(x)$. Using then the fact that

$$K := \sqrt{K_1^{(1)}} \le \liminf_{n\to\infty} \frac{\sigma(l_n/n, 1/2)}{(l_n/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(l_n/n)} \le \limsup_{n\to\infty} \frac{\sigma(l_n/n, 1/2)}{(l_n/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(l_n/n)} \le \sqrt{K_2^{(1)}}$$

by the first statement of Lemma 2.4(ii), for all n large enough we obtain

$$\begin{aligned}\Big|\varphi^{(1)}_{n,l_n}(x)\Big| &\le \frac{2}{K}\frac{\big(\frac{l_n}{n}\big)^{\frac{1}{\alpha}}}{\ell\big(\frac{l_n}{n}\big)}\left|\frac{\ell\big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\big)}{\big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\big)^{\frac{1}{\alpha}}}\Big[M_1\Big(\gamma\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big]\right. \\ &\qquad\left. - \frac{\ell\big(\frac{l_n}{n}\big)}{\big(\frac{l_n}{n}\big)^{\frac{1}{\alpha}}}\Big[M_1\Big(\gamma\Big(\frac{l_n}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n}\Big)\Big]\right| \\ &= \frac{2}{K}\left|u_n(x)\Big[M_1\Big(\gamma\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n}+x\frac{\sqrt{l_n}}{n}\Big)\Big] - \Big[M_1\Big(\gamma\Big(\frac{l_n}{n}\Big)\Big) + h_1\Big(\frac{l_n}{n}\Big)\Big]\right|,\end{aligned}$$

where

$$u_n(x) = \Big(1+\frac{x}{\sqrt{l_n}}\Big)^{-\frac{1}{\alpha}} \frac{\ell\big(\frac{l_n}{n}\big[1+\frac{x}{\sqrt{l_n}}\big]\big)}{\ell\big(\frac{l_n}{n}\big)} \to 1$$

by the slow variation of $\ell(\cdot)$ at zero. Since $M_1(\cdot)$ is bounded, we see, therefore, that the first convergence in (3.9) will follow if we show that

$$\big|\big[M_1(\gamma(t_n(x))) + h_1(t_n(x))\big] - \big[M_1(\gamma(s_n)) + h_1(s_n)\big]\big| \to 0, \tag{3.10}$$

where $s_n = l_n n^{-1} \to 0$ and $t_n(x) = l_n n^{-1} + x\sqrt{l_n}\, n^{-1} = s_n\big[1 + x l_n^{-1/2}\big] \to 0$. Since also, as a result of our continuity assumption, $\lim_{s\downarrow 0} h_1(s) = 0$ by the domain theorem at (2.23), and $t_n(0) = s_n$ of course, for (3.10) it suffices to show that

$$v_n(x) := \big|M_1(\gamma(t_n(x))) - M_1(\gamma(s_n))\big| \to 0 \quad \text{for each } x \neq 0. \tag{3.11}$$

Let $c \ge 1$ be the limit in (1.3c) for the sequence $\{k_n\}_{n=1}^{\infty}$ which defines $\gamma(\cdot)$ preceding (2.23). We may and do assume that $c > 1$ since in the case of $c = 1$, when $F \in \mathbb{D}(\alpha)$ for the given $\alpha \in (0,2)$ at hand and $M_1(\cdot)$ is a constant function, (3.11) is trivial. Then for all n large enough, $\gamma(s_n), \gamma(t_n(x)) \in [1, c^2]$, say, the definitions

$$\gamma_n(x) := \begin{cases} \gamma(t_n(x)), & \text{if } \gamma(t_n(x)) > \gamma(s_n), \\ c\,\gamma(t_n(x)), & \text{if } \gamma(t_n(x)) \le \gamma(s_n), \end{cases} \qquad \text{for } x > 0,$$

and

$$\gamma_n(x) := \begin{cases} \gamma(t_n(x)), & \text{if } \gamma(t_n(x)) < \gamma(s_n), \\ \frac{1}{c}\gamma(t_n(x)), & \text{if } \gamma(t_n(x)) \geq \gamma(s_n), \end{cases} \qquad \text{for } x < 0,$$

are meaningful and $c^{-1} \leq \gamma_n(x) \leq c^2$ for $x < 0$ and $1 \leq \gamma_n(x) \leq c^3$ for $x > 0$. Since $M_1(\gamma(t_n(x))) = M_1(\gamma_n(x))$ by the multiplicative periodicity of $M_1(\cdot)$, we have $v_n(x) = |M_1(\gamma_n(x)) - M_1(\gamma(s_n))|$ and, using the continuity condition for the second time, the function $M_1(\cdot)$ is uniformly continuous on the closed interval $[c^{-1}, c^3]$. Now, based on the definition of $\gamma(\cdot)$ above (2.23), the asymptotic equality

$$\frac{\gamma_n(x)}{\gamma(s_n)} \sim \frac{t_n(x)}{s_n}, \quad \text{where} \quad \frac{t_n(x)}{s_n} \to 1,$$

can be shown by enjoyable elementary arguments, which since the sequence $\{\gamma(s_n)\}$ is bounded, implies that $|\gamma_n(x) - \gamma(s_n)| \to 0$. The uniform continuity of $M_1(\cdot)$ then implies (3.11), proving the first statement in (3.9). Using the second statement of Lemma 2.4(ii), the proof of the second statement in (3.9) is completely analogous, and hence we have part (i) of the theorem.

Condition (3.5) for part (ii) of the theorem implies the existence of some finite positive constants $A_1 < 1 < A_2$ such that $A_1 m_n \leq l_n \leq A_2 m_n$ and $A_2^{-1} l_n \leq m_n \leq A_1^{-1} l_n$ for all n large enough. When proving (3.3), we renormalize $\varphi_{1,n}(\cdot)$ and $\varphi_{2,n}(\cdot)$ replacing $a_n(l_n, m_n)$ in the denominator by the sequences $a_n(A_1^{-1} l_n, A_1^{-1} l_n) \leq a_n(l_n, m_n)$ and $a_n(A_2 m_n, A_2 m_n) \leq a_n(l_n, m_n)$ to obtain the present versions of $\varphi^{(1)}_{n,l_n}(\cdot)$ and $\varphi^{(2)}_{n,m_n}(\cdot)$ of the proof above, respectively. For unified notation, we write $r_{1,n} = l_n$ and $r_{2,n} = m_n$.

Part (ii) itself has two cases. When $G = G_{0,0,\sigma}$ is normal for some $\sigma > 0$, we see by the criterion (1.26a) in Corollary 1 of [26] for the domain of attraction of a normal distribution that both terms in $\varphi^{(j)}_{n,r_{j,n}}(x)$ go to zero separately at every $x \in \mathbb{R}$, $j = 1, 2$, and hence (3.9) holds and implies (3.3) again.

Finally, the other case of part (ii) is when one of $M_1(\cdot)$ and $M_2(\cdot)$ in (2.10) and (2.23) is identically zero while the other is nowhere zero. Replacing K by $\sqrt{K_1}$ of part (i) of Lemma 2.4, the proof of (3.9) for that one of the present two sequences $\{\varphi^{(j)}_{n,r_{j,n}}(\cdot)\}$ for which $M_j(\cdot) > 0$, $j \in \{1, 2\}$, is practically the same as the one above for case (i), while it is simpler for the other $j \in \{1, 2\}$ for which $M_j(\cdot) = 0$ because (3.10) for that $M_j(\cdot)$ is trivial. Thus condition (3.3) for asymptotic normality holds true once more. ■

Besides Lemma 2.4 the proofs of Theorem 3.2 and Theorem 3.3 also require the following lemma.

Lemma 3.1. *Suppose that $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha, \psi_2^\alpha, 0})$ with a quantile function given by (2.23). Then for any $a, b \in [1, c)$ that are continuity points of both M_j, $j = 1, 2$, and for any $\delta > 0$ and $\varepsilon \in (0, 1)$ there exists a threshold number $N(a, b, \delta, \varepsilon)$ such that the inequality*

$$\begin{aligned} |M^*_{n,j}(y_1) - M^*_{n,j}(y_2)| &= \left| \left[M_j\left(\gamma\left(\frac{y_1}{k_n}\right)\right) + h_j\left(\frac{y_1}{k_n}\right) \right] - \left[M_j\left(\gamma\left(\frac{y_2}{k_n}\right)\right) + h_j\left(\frac{y_2}{k_n}\right) \right] \right| \\ &\leq 2\left[|M_j(a) - M_j(b)| + C(a, b, \alpha, \varepsilon) + \delta \right] \end{aligned} \tag{3.12}$$

holds true for all $n \ge N(a,b,\delta,\varepsilon)$ *and* $y_1, y_2 \in [a,b]$, $j = 1,2$, *where*

$$C(a,b,\alpha,\varepsilon) = \max\{D_1, D_2\}\left[1 - (1-\varepsilon)\left(\frac{a}{b}\right)^{1/\alpha}\right]$$

with the constants D_1 *and* D_2 *from (2.41) and* $M^*_{n,j}(\cdot) = M_j(\gamma(\cdot/k_n)) + h_j(\cdot)$, $j = 1,2$.

Proof. Notice first that (3.12) is trivial if $M_j \equiv 0$. Thus, since the half-sided version of the proof below will be an obvious special case when exactly one $M_j \equiv 0$, it suffices to deal with the situation when $M_j \not\equiv 0$, $j = 1,2$. In this situation M_1 and M_2 both have positive infima on $(0,\infty)$ and we see by applying (2.23) for $s = t/k_n$, where $t > 0$ is a continuity point of M_1 and M_2, and by the monotone nondecreasing nature of Q that $\lim_{s\downarrow 0} Q(s) = -\infty$ and $\lim_{s\uparrow 1} Q(s) = \infty$. We choose $N^* = N(a,b,\delta,\varepsilon)$ so that

$$Q_+\left(\frac{b}{k_n}\right) < 0 \quad \text{and} \quad Q\left(1 - \frac{b}{k_n}\right) > 0, \quad \text{and so} \quad M^*_{n,j}(y) > 0, \;\; a \le y \le b, \;\; j = 1,2,$$

$$\gamma(a/k_n) = a \quad \text{and} \quad \gamma(b/k_n) = b,$$

$$\left|1 - \frac{\ell(y_1/k_n)}{\ell(y_2/k_n)}\right| < \varepsilon, \;\; y_1, y_2 \in [a,b], \quad \text{and} \quad \left|h_j\left(\frac{a}{k_n}\right)\right| + \left|h_j\left(\frac{b}{k_n}\right)\right| < \delta, \;\; j = 1,2,$$

hold simultaneously whenever $n \ge N^*$ and show (3.12) with this choice of the threshold.

Assuming without loss of generality that $a \le y_1 \le y_2 \le b$, notice that

$$\frac{Q_+\left(\frac{y_1}{k_n}\right)}{Q_+\left(\frac{y_2}{k_n}\right)} \ge 1 \quad \text{and} \quad \frac{Q\left(1 - \frac{y_1}{k_n}\right)}{Q\left(1 - \frac{y_2}{k_n}\right)} \ge 1, \quad \text{that is,} \quad \frac{\left(\frac{y_1}{k_n}\right)^{-\frac{1}{\alpha}} \ell\left(\frac{y_1}{k_n}\right) M^*_{n,j}(y_1)}{\left(\frac{y_2}{k_n}\right)^{-\frac{1}{\alpha}} \ell\left(\frac{y_2}{k_n}\right) M^*_{n,j}(y_2)} \ge 1, \;\; j = 1,2,$$

and so

$$\frac{M^*_{n,j}(y_1)}{M^*_{n,j}(y_2)} \ge \left(\frac{y_1}{y_2}\right)^{1/\alpha} \frac{\ell(y_1/k_n)}{\ell(y_2/k_n)} \ge \left(\frac{a}{b}\right)^{1/\alpha} (1-\varepsilon), \;\; j = 1,2,$$

if $n \ge N^*$. Recalling (2.41) we see that for $n \ge N^*$ the inequality

$$\left|M^*_{n,j}(y_1) - M^*_{n,j}(y_2)\right| \le C(a,b,\alpha,\varepsilon) \tag{3.13}$$

holds true for any choice of $y_1, y_2 \in [a,b]$, $y_1 \le y_2$, provided $M^*_{n,j}(y_1) \le M^*_{n,j}(y_2)$, $j = 1,2$. If this is the case indeed, then (3.13) in itself proves (3.12), but the following considerations apply in general. Indeed, observe that the choice of N^* ensures that

$$\left|M^*_{n,j}(a) - M^*_{n,j}(b)\right| < \left|M_j(a) - M_j(b)\right| + \delta, \quad j = 1,2, \tag{3.14}$$

for all $n \ge N^*$, and note also that

$$\left|M^*_{n,j}(y_1) - M^*_{n,j}(y_2)\right| \le \left|M^*_{n,j}(a) - M^*_{n,j}(y_1)\right| + \left|M^*_{n,j}(a) - M^*_{n,j}(y_2)\right|.$$

Here $|M^*_{n,j}(a) - M^*_{n,j}(y_l)| \le C(a,b,\alpha,\varepsilon)$ by (3.13) if $M^*_{n,j}(a) \le M^*_{n,j}(y_l)$, $l,j \in \{1,2\}$, but if this fails for some $l,j \in \{1,2\}$ then we still have

$$\left|M^*_{n,j}(a) - M^*_{n,j}(y_l)\right| \le \left|M^*_{n,j}(a) - M^*_{n,j}(b)\right| + \left|M^*_{n,j}(b) - M^*_{n,j}(y_l)\right|,$$

where the first term can be estimated using (3.14) and $C(a,b,\alpha,\varepsilon)$ is an upper bound on the second one, provided $M^*_{n,j}(y_l) \le M^*_{n,j}(b)$. However, if the latter is not the case either, then $|M^*_{n,j}(a) - M^*_{n,j}(y_l)| \le |M^*_{n,j}(a) - M^*_{n,j}(b)|$, since $M^*_{n,j}(a) > M^*_{n,j}(y_l) > M^*_{n,j}(b)$. All this together imply (3.12). ■

Proof of Theorem 3.2. We only have to deal with the case when $G = G_{\psi_1^\alpha,\psi_2^\alpha,0}$ for some $\alpha \in (0,2)$, where at least one of ψ_1^α and ψ_2^α is not identically 0. The other case being analogous, suppose that $\psi_1^\alpha \not\equiv 0$. Retaining the notation in the proof of Theorem 3.1, we show that a sequence $\{l_n\}$ of positive integers can be chosen to satisfy both (3.1) and the first convergence relation in (3.9). The latter follows, through the same considerations as there, if $\{l_n\}$ is chosen to make sure that (3.10) holds.

By the monotonicity of ψ_1^α we can pick a sequence of pairs (a_j, b_j), $1 \le a_j < b_j < c$, and constants $\varepsilon_j \in (0,1)$ such that both a_j and b_j are continuity points of M_1 and the inequalities $|M_1(a_j) - M_1(b_j)| < \frac{1}{6j}$ and $C(a_j, b_j, \alpha, \varepsilon_j) < \frac{1}{6j}$ hold for all $j \in \mathbb{N}$. Next, put $N_0^\circ := 0$ and, by means of the threshold numbers of Lemma 3.1, define an increasing sequence $\{N_j^\circ\}$ of positive integers by setting $N_j^\circ := \max\{N(a_j, b_j, \frac{1}{6j}, \varepsilon_j), N_{j-1}^\circ + 1\}$, $j \in \mathbb{N}$. Elementary consideration shows now that for each $j \in \mathbb{N}$ there exists a threshold number $N_j^* \in \mathbb{N}$ such that for every $n \ge N_j^*$ one can choose an $l^*_{n,j} \in \mathbb{N}$ with the properties that

$$\left[\frac{l^*_{n,j}}{n} - j\frac{\sqrt{l^*_{n,j}}}{n}, \frac{l^*_{n,j}}{n} + j\frac{\sqrt{l^*_{n,j}}}{n}\right] \subset \left[\frac{a_j}{k_{N_j^\circ}}, \frac{b_j}{k_{N_j^\circ}}\right] \quad \text{and} \quad l^*_{n+1,j} \ge l^*_{n,j}, \tag{3.15}$$

and it can clearly be stipulated that $1 < N_1^* < N_2^* < \cdots$. By Lemma 3.1 we see that

$$\left|\left[M_1\left(\gamma\left(\frac{l^*_{n,j}}{n} + x\frac{\sqrt{l^*_{n,j}}}{n}\right)\right) + h_1\left(\frac{l^*_{n,j}}{n} + x\frac{\sqrt{l^*_{n,j}}}{n}\right)\right] - \left[M_1\left(\gamma\left(\frac{l^*_{n,j}}{n}\right)\right) + h_1\left(\frac{l^*_{n,j}}{n}\right)\right]\right| \le \frac{1}{j}$$

for all $x \in [-j,j]$ and $n \ge N_j^*$.

Now we are ready to choose the desired sequence $\{l_n\}$. We set $l_n := 1$ for $n < N_1^*$ and define $\{l_n\}_{n=N_1^*}^\infty$ by the following algorithm, in which $T \in \mathbb{N}$ is a new auxiliary variable:

Step 1. Let the initial values of j and n be $j := 1$ and $n := N_1^*$, and put $T := N_1^*$.

Step 2. If $N_j^* \le n < N_{j+1}^*$ then let $l_n := l^*_{n,j}$.

Step 3. If $n \ge N_{j+1}^*$ then put $l_n := l^*_{n,j}$ or $l_n := l^*_{n,j+1}$ according as $l^*_{n,j+1} \le l_T$ or $l^*_{n,j+1} > l_T$, and if $l^*_{n,j+1} > l_T$ then set also $j := j+1$ and $T := n$.

Step 4. Set $n := n+1$ and go to Step 2.

Then $l_n \to \infty$ by the choices of T and, since $N_j^\circ \to \infty$ as $j \to \infty$, we also have $l_n/n \to 0$ by (3.15). Thus (3.1) holds for the chosen sequence $\{l_n\}$ and the displayed inequality following (3.15) above shows that (3.10) is also satisfied for any fixed $x \in \mathbb{R}$.

If $\psi_2^\alpha \not\equiv 0$, then the sequence $\{m_n\}$ can be chosen in a similar fashion. If $\psi_2^\alpha \equiv 0$, then simply put $m_n := l_n$ for every $n \in \mathbb{N}$, and the desired asymptotic normality follows as in the proof of part (ii) of Theorem 3.1. ∎

Lemma 3.2. *If a function $\ell(\cdot)$ on $(0,1)$ is slowly varying at zero, $F \in \mathbb{D}_{\rm gp}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ for a semistable $G_{\psi_1^\alpha,\psi_2^\alpha,0}$ of exponent $\alpha \in (0,2)$, and $\{r_n\}$ is a sequence of positive numbers such that $r_n \to \infty$ and $r_n/n \to 0$, then*

$$\frac{(r_n/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(r_n/n)}{(1/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(1/n)} \to 0 \qquad \textit{and} \qquad \frac{\sigma\left(\frac{r_n}{n}, 1-\frac{r_n}{n}\right)}{(1/n)^{\frac{1}{2}-\frac{1}{\alpha}}\ell(1/n)} \to 0.$$

Proof. The first statement is just a special case of Lemma 2 in [30], while the second follows from the first and part (i) of Lemma 2.4. ∎

Proof of Theorem 3.3. We see by (2.41) and the first statement of Lemma 3.2 that, which in actual fact takes place along the whole sequence $\{n\} = \mathbb{N}$,

$$\frac{\sqrt{l_{k_n}}\, Q_+\!\left(\frac{l_{k_n}}{k_n} + x\frac{\sqrt{l_{k_n}}}{k_n}\right)}{k_n^{1/\alpha}\ell(1/k_n)} \to 0 \quad \text{and} \quad \frac{\sqrt{m_{k_n}}\, Q\!\left(1-\frac{m_{k_n}}{k_n} + x\frac{\sqrt{m_{k_n}}}{k_n}\right)}{k_n^{1/\alpha}\ell(1/k_n)} \to 0 \quad \text{for all } x \in \mathbb{R}.$$

Next, by the proof of the sufficiency part of Theorem 2.3, we have

$$\frac{Q_+\!\left(\frac{y}{k_n}\right)}{k_n^{1/\alpha}\ell(1/k_n)} \to \psi_1^\alpha(y) \quad \text{and} \quad \frac{Q\!\left(1-\frac{y}{k_n}\right)}{k_n^{1/\alpha}\ell(1/k_n)} \to \psi_2^\alpha(y) \tag{3.16}$$

at all the respective continuity points $y > 0$ of the limiting functions. Furthermore, Lemma 2.4(i) implies that

$$\limsup_{n\to\infty} \frac{\sqrt{k_n}\,\sigma\!\left(\frac{1}{k_n}, \frac{l_{k_n}}{k_n}\right)}{k_n^{1/\alpha}\ell(1/k_n)} \le \sqrt{K_2} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\sqrt{k_n}\,\sigma\!\left(1-\frac{m_{k_n}}{k_n}, 1-\frac{1}{k_n}\right)}{k_n^{1/\alpha}\ell(1/k_n)} \le \sqrt{K_2}.$$

Finally, also along the whole sequence $\{n\} = \mathbb{N}$ in fact, Lemma 3.2 implies

$$\frac{\sqrt{k_n}\,\sigma\!\left(\frac{r_{k_n}}{k_n}, \frac{l_{k_n}}{k_n}\right)}{k_n^{1/\alpha}\ell(1/k_n)} \to 0 \quad \text{and} \quad \frac{\sqrt{k_n}\,\sigma\!\left(1-\frac{m_{k_n}}{k_n}, 1-\frac{r_{k_n}}{k_n}\right)}{k_n^{1/\alpha}\ell(1/k_n)} \to 0$$

for any sequence $\{r_n\}$ of positive numbers such that $r_n \to \infty$, $r_n/l_n \to 0$ and $r_n/m_n \to 0$.

These four pairs of facts allow a subsequential application of that variant of a two-sided version of Theorem 1 in [28], the version alluded to on p. 789 there, in which the basic functions $Q_+(s)$ and $Q(1-s)$, $0 < s < 1$, are taken right-continuous and the Poisson processes $N_1(\cdot)$ and $N_2(\cdot)$ are taken left-continuous as in the present paper. Using the eight facts above, this variant implies that every subsequence of $\mathbb{N}$ contains a further subsequence such that (3.6) and (3.8) hold jointly along that subsequence. This implies that (3.6) and (3.8) hold jointly as stated.

By the convergence of types theorem, (3.6) and (3.8) already imply (3.7) for the subsequence $\{k_n\}$. However, if neither of M_1 and M_2, or equivalently, neither of ψ_1^α and ψ_2^α is identically zero, then the left side of (3.10) is bounded, by $2(D_1+D_2)$ from (2.41), for both M_1 and M_2 even if they are not continuous, implying that the two sequences of functions in (3.9) are pointwise bounded. Hence the same is true for the sequences $\{\varphi_{j,n}(\cdot)\}$, $j \in \{1,2\}$. Also, setting $r_n \equiv \min(l_n, m_n)$, we have $a_n(l_n, m_n) \le a_n(r_n, r_n)$ for all $n \in \mathbb{N}$ and $a_n(r_n, r_n)/[n^{1/\alpha}\ell(1/n)] \to 0$ by Lemma 3.2. Therefore, the discussion at (1.13) in [27] yields (3.7) as stated.

If, on the other hand, $M_1 \equiv 0$ and $M_2 \not\equiv 0$, then by the same argument

$$\frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=m_n+1}^{n-m_n} X_{j,n} - n\int_{\frac{m_n}{n}}^{1-\frac{m_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0$$

and, since in this case the first convergence in (3.16) takes place along the whole $\{n\} = \mathbb{N}$ with an identically zero limiting function, we also get

$$\frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=1}^{r_n} X_{j,n} - n\int_{\frac{1}{n}}^{\frac{r_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0 \quad \text{for both } r_n \equiv l_n \text{ and } r_n \equiv m_n,$$

which together prove (3.7).

We see that if $M_1(\cdot) \equiv 0$ and $M_2(\cdot) > 0$, then in fact we have

$$\frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=1}^{n-m_n} X_{j,n} - n\int_{\frac{1}{n}}^{1-\frac{m_n}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0$$

along with (3.8). Similarly, if $M_2(\cdot) \equiv 0$ and $M_1(\cdot) > 0$, then again we have (3.7) and, in fact,

$$\frac{1}{n^{1/\alpha}\ell(1/n)}\left\{\sum_{j=l_n+1}^{n} X_{j,n} - n\int_{\frac{l_n}{n}}^{1-\frac{1}{n}} Q(u)\,du\right\} \xrightarrow{\mathbb{P}} 0$$

along with (3.6). ∎

4. Max-semistable laws

4.1. Introduction

As before, let $X_1, X_2, \dots$ be a sequence of independent identically distributed random variables with a common distribution function $F(x) := \mathbb{P}\{X_1 \le x\}$, $x \in \mathbb{R}$, and let $M_n := \max\{X_1, \dots, X_n\}$ denote their maximum. The possible limiting distributions of the sequence $b_n^{-1}(M_n - a_n)$, where $b_n > 0$, were determined by Gnedenko [43] in the early forties (see also the monographs de Haan [50], Galambos [42] and Resnick [68]) — these distributions are called max-stable. If converge in distribution takes place only along subsequences $\{k_n\}$ that satisfy the geometric growth condition in (1.3c), the arising limiting laws are exactly the max-semistable laws. Thus, the conditions are that

$$\mathbb{P}(b_n^{-1}(M_{k_n} - a_n) \le x) = \Big(F\big(b_n x + a_n\big)\Big)^{k_n} \to G(x), \tag{4.1}$$

for all $x \in C_G$, where C_G denotes the set of continuity points of the distribution function G, and the eventually strictly increasing subsequence $\{k_n\}$ satisfies (1.3c), that is,

$$\frac{k_{n+1}}{k_n} \to c \ge 1.$$

Note that the notation used for the centering and norming sequences differs from the one used for sums in Section 2 and 3. If (4.1) holds we say that F is in the max-domain of geometric partial attraction of the max-semistable law G, written $F \in \mathbb{MD}_{\mathrm{gp}}(G)$. For the sake of simplicity, we will always assume $c > 1$ unless explicitly stated otherwise, since for $c = 1$ the limiting law is max-stable (see Pancheva [65] or Theorem 4.7 below). If these conditions hold then the centering and norming constants necessarily satisfy

$$\frac{b_{n+1}}{b_n} \to \gamma \qquad \text{and} \qquad \frac{a_{n+1} - a_n}{b_n} \to \beta \tag{4.2}$$

where $\gamma > 0$, $\beta \in \mathbb{R}$ and the limiting distribution G is one of the three below:

$$\begin{aligned}
G(x) &= \Lambda_{\alpha,\nu}(x) := \exp\big(-\exp(-\alpha x)\nu(x)\big), & x \in \mathbb{R},\\
G(x) &= \Phi_{\alpha,\nu,t}(x) := \exp\Big(-(x-t)^{-\alpha}\nu\big(\log(x-t)\big)\Big), & x \in (t,\infty),\\
G(x) &= \Psi_{\alpha,\nu,t}(x) := \exp\Big(-(t-x)^{\alpha}\nu\big(\log(t-x)\big)\Big), & x \in (-\infty,t).
\end{aligned}$$

In the first case (assuming $c > 1$) necessarily

$$\frac{b_{n+1}}{b_n} \to 1, \qquad \frac{a_{n+1} - a_n}{b_n} \to \beta > 0. \tag{4.3}$$

The constant α is given by $\alpha = \frac{\log c}{\beta}$ and the positive bounded function $\nu(\cdot)$ is periodic with period β, satisfying the monotonicity condition that

$$\exp(-\alpha x)\nu(x) \quad \text{is non-increasing, } x \in \mathbb{R}. \tag{M0}$$

However, the constant α here (unlike in the next two cases) is only a scale parameter: by the Convergence of Types Theorem one can always achieve $\alpha = 1$. We may and do assume this without loss of generality. In this case $\beta = \log c$ and condition (M0) is valid with $\alpha = 1$. The limiting law is of the form:

$$\Lambda_\nu(x) := \Lambda_{1,\nu}(x) = \exp\big(-\exp(-x)\nu(x)\big), \quad x \in \mathbb{R}. \tag{4.4}$$

In the second and third cases, $t \in \mathbb{R}$ is determined by $\beta = t(1-\gamma)$ and we necessarily have $\gamma > 1$ in the second and $\gamma < 1$ in the third case. The constant α is given by $\alpha = \left|\frac{\log c}{\log \gamma}\right|$ and the positive bounded functions $\nu(\cdot)$ are periodic with period $|\log \gamma|$, satisfying the corresponding monotonicity conditions that

$$x^{-\alpha}\nu(\log x) \text{ is non-increasing for } x > 0; \tag{$M-$}$$

$$x^{\alpha}\nu(\log x) \text{ is non-decreasing for } x > 0. \tag{$M+$}$$

It was Grinevich and Pancheva who first described max-semistable laws: all the results and statements above can be found in Grinevich [46] and Grinevich [47]. Using the Convergence of Types Theorem again one can always achieve $t = 0$ in case of $\Phi_{\alpha,\nu,t}$ and $\Psi_{\alpha,\nu,t}$: neither the type of the limiting distribution nor the set of the attracted distributions will be changed. So we may and do assume without loss of generality that this is the case: then (4.2) can be rewritten as

$$\frac{b_{n+1}}{b_n} \to \gamma > 0, \qquad \frac{a_{n+1} - a_n}{b_n} \to 0, \tag{4.5}$$

where $\gamma > 1$ if $G = \Phi_{\alpha,\nu,t}$ and $\gamma < 1$ if $G = \Psi_{\alpha,\nu,t}$, and the limiting laws are of the form

$$\Phi_{\alpha,\nu}(x) := \Phi_{\alpha,\nu,0} = \exp\big(-x^{-\alpha}\nu(\log x)\big), \qquad x \in (0,\infty), \tag{4.6}$$

$$\Psi_{\alpha,\nu}(x) := \Psi_{\alpha,\nu,0} = \exp\Big(-|x|^{\alpha}\nu\big(\log|x|\big)\Big), \qquad x \in (-\infty, 0). \tag{4.7}$$

If $c = 1$ then it is easily seen that $\gamma = 1$ and $\beta = 1$ and hence $\alpha = \left|\frac{\log c}{\log \gamma}\right|$ is undefined. Consequently, it cannot be used to determine the characteristic exponent of the necessarily max-stable limiting law. In this case one should pass to a subsequence $\{k_{n(r)}\}_{r=1}^{\infty}$ such that $k_{n(r+1)}/k_{n(r)} \to c' > 1$ and use the asymptotic behaviour of the centering and norming sequences along $\{k_{n(r)}\}$.

4.2. The Characterization of Max-Domains of Geometric Partial Attraction

Denote the right endpoint of the support of a distribution F by $\operatorname{rext} F := \sup\{x : F(x) < 1\} \in (-\infty, \infty]$ and observe that (4.1) implies

$$b_n x + a_n \to \operatorname{rext} F \qquad \text{for all } x \text{ such that } 0 < G(x) < 1. \tag{4.8}$$

First we want to give a description for $\mathbb{MD}_{\text{gp}}(\Phi_{\alpha,\nu})$ and $\mathbb{MD}_{\text{gp}}(\Psi_{\alpha,\nu})$. It is well known (see Gnedenko [43] or de Haan [50]) that $F \in \mathbb{MD}(\Phi_{\alpha,1(\cdot)}) = \mathbb{MD}(\exp(-x^{-\alpha}))$ or $F \in \mathbb{MD}(\Psi_{\alpha,1(\cdot)}) = \mathbb{MD}(\exp(-|x|^{\alpha}))$ if and only if $1 - F(x)$ or $1 - F(\operatorname{rext} F - x^{-1})$, respectively, is regularly varying as $x \to \infty$ with index $-\alpha$. Here $\mathbb{MD}(S)$ denotes the max-domain of attraction of the max-stable law S, i.e., the set of F's for which (4.1) holds with $k_n \equiv n$. It is not surprising that both $\mathbb{MD}_{\text{gp}}(\Phi_{\alpha,\nu})$ and $\mathbb{MD}_{\text{gp}}(\Psi_{\alpha,\nu})$ are related to regular variation and can be characterized in a rather similar manner.

Theorem 4.1. *Assume $F \in \mathbb{MD}_{\text{gp}}(\Phi_{\alpha,\nu})$ with centering and norming constants a_n and b_n. Then (4.5) and (4.8) hold and necessarily*

$$\frac{a_n}{b_n} \to 0. \tag{4.9}$$

Furthermore, there exists a distribution function F_0 such that
(i) $\operatorname{rext} F_0 = \operatorname{rext} F = \infty$ and $1 - F_0(\cdot)$ is regularly varying at ∞ with index $-\alpha$;
(ii) introducing $\theta(x) := \big(1 - F(x)\big)/\big(1 - F_0(x)\big)$ and fixing some $x_* \in C_{\nu(\log(\cdot))} = C_{\Phi_{\alpha,\nu}} \cap (0, \infty)$, we have

$$\frac{\theta(b_n x + a_n)}{\theta(b_n x_* + a_n)} \to \frac{\nu(\log x)}{\nu(\log x_*)} \tag{4.10}$$

for each $x \in C_{\nu(\log(\cdot))} = C_{\Phi_{\alpha,\nu}} \cap (0, \infty)$.

In addition, the centering sequence $\{a_n\}$ is negligible, meaning that

$$\Big(F\big(b_n x + a_n\big)\Big)^{k_n} \to \Phi_{\alpha,\nu}(x), \qquad x \in C_{\Phi_{\alpha,\nu}},$$

implies

$$\Big(F\big(b_n x\big)\Big)^{k_n} \to \Phi_{\alpha,\nu}(x), \qquad x \in C_{\Phi_{\alpha,\nu}},$$

and so (4.10) holds with $a_n \equiv 0$ as well.

Conversely, if (4.10) holds with a sequence of norming and centering constants $b_n > 0$ and $a_n \in \mathbb{R}$ satisfying (4.5), (4.8) and (4.9) then there exists a sequence k_n satisfying (1.3c),

$$k_n = \left\lfloor \frac{\nu(\log x_*) x_*^{-\alpha}}{1 - F(b_n x_* + a_n)} \right\rfloor \tag{4.11}$$

for example, such that (4.1) holds with $G = \Phi_{\alpha,\nu}$, hence $F \in \mathbb{MD}_{\text{gp}}(\Phi_{\alpha,\nu})$.

The conditions of the theorem are necessary and sufficient, thus $F \in \mathbb{MD}_{\text{gp}}(\Phi_{\alpha,\nu})$ is completely characterized now. Of course, if $1 \in C_{\nu(\log(\cdot))}$ then it is convenient to choose $x_* = 1$ in (4.10). The proof of the next theorem goes along the same lines as that of Theorem 4.1, hence omitted.

Theorem 4.2. *Assume $F \in \mathbb{MD}_{\text{gp}}(\Psi_{\alpha,\nu})$ with centering and norming constants a_n and b_n. Then (4.5) and (4.8) hold,* $\operatorname{rext} F < \infty$ *and necessarily*

$$\frac{\operatorname{rext} F - a_n}{b_n} \to 0. \tag{4.12}$$

Furthermore, there exists a distribution function F_0 such that

(i) $\operatorname{rext} F_0 = \operatorname{rext} F < \infty$ *and $1 - F_0(\operatorname{rext} F_0 - x^{-1})$ is regularly varying as $x \to \infty$ with index $-\alpha$;*

(ii) introducing $\theta(x) := (1-F(x))/(1-F_0(x))$ and fixing some $x_* \in C_{\nu(\log|\cdot|)} = C_{\Psi_{\alpha,\nu}} \cap (-\infty, 0)$, we have

$$\frac{\theta(b_n x + a_n)}{\theta(b_n x_* + a_n)} \to \frac{\nu(\log|x|)}{\nu(\log|x_*|)} \tag{4.13}$$

for each $x \in C_{\nu(\log|\cdot|)} = C_{\Psi_{\alpha,\nu}} \cap (-\infty, 0)$.

In addition, one may always choose $a_n \equiv \operatorname{rext} F$:

$$\Big(F\big(b_n x + a_n\big)\Big)^{k_n} \to \Psi_{\alpha,\nu}(x), \qquad x \in C_{\Psi_{\alpha,\nu}},$$

implies

$$\Big(F\big(b_n x + \operatorname{rext} F\big)\Big)^{k_n} \to \Psi_{\alpha,\nu}(x), \qquad x \in C_{\Psi_{\alpha,\nu}},$$

and hence (4.13) holds with $a_n \equiv \operatorname{rext} F$ as well.

Conversely, if (4.13) holds with a sequence of norming and centering constants $b_n > 0$ and $a_n \in \mathbb{R}$ satisfying (4.5), (4.8) and (4.12) then there exists a sequence k_n satisfying (1.3c),

$$k_n = \left\lfloor \frac{\nu(\log|x_*|)|x_*|^\alpha}{1 - F(b_n x_* + a_n)} \right\rfloor$$

for example, such that (4.1) holds with $G = \Psi_{\alpha,\nu}$, hence $F \in \mathbb{MD}_{\text{gp}}(\Psi_{\alpha,\nu})$.

Theorems 4.1 and 4.2 are essentially the corrected versions of Theorem 2.1 in Grinevich [48]. The main difference between these two theorems and the one in Grinevich [48] is in the sufficiency parts. In the Russian version of Grinevich [48] it is explicitly stated, and comparing equations (2.9) and (2.10) in the English version still follows that if $F \in \mathbb{MD}_{\text{gp}}(\Phi_{\alpha,\nu})$ (or $F \in \mathbb{MD}_{\text{gp}}(\Psi_{\alpha,\nu})$) then $F \in \mathbb{MD}_{\text{gp}}(\Phi_{\alpha,\nu})$ (or $F \in \mathbb{MD}_{\text{gp}}(\Psi_{\alpha,\nu})$) along $k_n = \lfloor \nu(0) n c^n \rfloor$. However, this is false in general, as it will be shown by the Counterexample below. In fact, there is no fixed k_n such that $F \in \mathbb{MD}_{\text{gp}}(G)$ with a semistable limiting distribution G implies convergence along k_n. It will be shown in Section 3 where a detailed analysis of the structure of max-domains of geometric partial attraction is given that if we have convergence along a subsequence k_n then, roughly speaking, the only subsequences along which convergence takes place are k_n itself and the ones 'proportional' to it (for the precise statement see Theorem 4.6). The reason that Grinevich reaches the false conclusion that $F \in \mathbb{MD}_{\mathrm{gp}}(\Phi_{\alpha,\nu})$ (or $F \in \mathbb{MD}_{\mathrm{gp}}(\Psi_{\alpha,\nu})$) implies that $F \in \mathbb{MD}_{\mathrm{gp}}(\Phi_{\alpha,\nu})$ (or $F \in \mathbb{MD}_{\mathrm{gp}}(\Psi_{\alpha,\nu})$) along $k_n = \lfloor \nu(0) n c^n \rfloor$ is because she requires (4.10) (or (4.13)) to hold, which formulas already involve b_n and a_n, and *then* she chooses b_n and a_n, which is illegal. The Counterexample shows that the sufficiency part of Theorem 2.1 in Grinevich [48] fails for $F \in \mathbb{MD}_{\mathrm{gp}}(\Phi_{\alpha,\nu})$: it is not difficult to construct another counterexample to show that it fails in case of $F \in \mathbb{MD}_{\mathrm{gp}}(\Psi_{\alpha,\nu})$ just as well.

Counterexample *(The St. Petersburg game).* Recall from (1.4) the distribution function

$$F(x) := 1 - 2^{-\lfloor \mathrm{Log} x \rfloor}, \qquad x \geq 2,$$

with Log standing for the logarithm to the base 2. Define

$$H_\gamma(x) := \begin{cases} 0, & x \leq 0, \\ \exp(-\gamma 2^{-\lfloor \mathrm{Log}(\gamma x) \rfloor}), & x > 0, \end{cases}$$

and $\gamma_n := n/2^{\lceil \mathrm{Log} n \rceil} \in (1/2, 1]$, where $\lceil y \rceil = \min\{k \in \mathbb{N} : k \geq y\}$ denotes the upper integer part of $y > 0$. Clearly, $H_\gamma = \Phi_{1,\nu_\gamma}$ with $\nu_\gamma(\log x) = 2^{\mathrm{Log}(\gamma x) - \lfloor \mathrm{Log}(\gamma x) \rfloor}$. It can be seen by an application of Theorem 3.3 or directly from relation (4) in Berkes, Csáki, and Csörgő [9] that

$$\sup_{-\infty < x < \infty} \left| \mathbb{P}\left\{ \frac{M_n}{n} \leq x \right\} - H_{\gamma_n}(x) \right| \to 0. \tag{4.14}$$

Since the distribution functions H_γ are of different type for $\gamma \in (1/2, 1]$, this implies that if k_n is any increasing sequence of positive integers, then M_{k_n}/k_n has a limiting distribution if and only if the sequence γ_n converges to some $\gamma \in [1/2, 1]$ or has exactly two limit points, $1/2$ and 1. For example, we have weak convergence along $k_n = 2^n$ or $\lfloor d 2^n \rfloor$ for $d > 0$, but the sequence M_{k_n}/k_n does not converge weakly along $k_n = \lfloor n 2^n d \rfloor$ for any $d > 0$. By the necessity part of Theorem 4.1 above, neither does $[M_{k_n} - a_n]/k_n$ for any admissible choice of $\{a_n\}$. Using the Convergence of Types Theorem, this in turn implies that $F(b_n x + a_n)^{\lfloor n 2^n d \rfloor}$ does not converge weakly to any non-degenerate distribution for any choice of $\{a_n\}$, $\{b_n\}$ and $d > 0$. This contradicts the sufficiency part of Theorem 2.1 in Grinevich [48].

Applying Theorem 2.7 along $\{k_n\}$ it follows that if $F \in \mathbb{D}_{\mathrm{gp}}(G)$ with a semistable G (the notion 'semistable' being used as in Sections 1–3, in the sense of sums) then M_n also converges along $\{k_n\}$ to some non-degenerate limiting law $K(\cdot)$ (determined by G, of course) with the normalization used for the full sum and with centralization $a_n \equiv 0$. Hence, if $F \in \mathbb{D}_{\mathrm{gp}}(G)$ then $F \in \mathbb{MD}_{\mathrm{gp}}(K)$, that is, F belongs to the max-domain of geometric partial attraction of a max-semistable law. If $F \in \mathbb{D}_{\mathrm{gp}}(G)$ then the K determined by G is necessarily of the form $\Phi_{\alpha,\nu}$ with a suitable $\nu(\cdot)$ and $\alpha \in (0, 2)$. Theorem 2.7 now states the maxima of independent identically distributed random variables with common distribution function from a subset of $\mathbb{MD}_{\mathrm{gp}}(\Phi_{\alpha,\nu})$ merge together with a family of limiting distributions. One of our main goals now is to show that the merge of maxima extends to the whole $\mathbb{MD}_{\mathrm{gp}}(\Phi_{\alpha,\nu})$, $\mathbb{MD}_{\mathrm{gp}}(\Psi_{\alpha,\nu})$ and $\mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$. However, still we have to give a characterization for $\mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$. To this end we need the notion of a von Mises function, which is defined as follows (see Resnick [68]).

Definition. *A distribution function $F_\sharp$ is called a von Mises function with auxiliary function $f(\cdot)$ if there exists $z_0 < \operatorname{rext} F_\sharp$ such that for $x \in (z_0, \operatorname{rext} F_\sharp)$*

$$1 - F_\sharp(x) = d \exp\left\{ - \int_{z_0}^{x} \frac{1}{f(u)}\, du \right\},$$

where $d > 0$, $f(u) > 0$ for $u \in (z_0, \operatorname{rext} F_\sharp)$ and $f(\cdot)$ is absolutely continuous on $(z_0, \operatorname{rext} F_\sharp)$ with density f' and $f'(u) \to 0$ as $u \uparrow \operatorname{rext} F_\sharp$.

It is well known (see de Haan [50]) that $F \in \mathbb{MD}(\Lambda_{1(\cdot)})$, where $1(x) = 1$, $x \in \mathbb{R}$, and thus $\Lambda_{1(\cdot)} = \exp(-\exp(-x))$, the Gumbel law, if and only if there exists a von Mises function $F_\sharp$ such that $1 - F(x) \sim 1 - F_\sharp(x)$ as $x \uparrow \operatorname{rext} F$.

Theorem 4.3. *Assume $F \in \mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$ with centering and norming constants a_n and b_n. Then (4.3) is satisfied and (4.8) holds for all $x \in \mathbb{R}$, and, furthermore,*

(i) there exists a von Mises function $F_\sharp$ with auxiliary function $f(\cdot)$ for which $\operatorname{rext} F_\sharp = \operatorname{rext} F$ and

$$b_n = f(a_n) \tag{4.15}$$

for all $n \in \mathbb{N}$ large enough;

(ii) introducing $\theta(x) := \big(1 - F(x)\big)/\big(1 - F_\sharp(x)\big)$ and fixing some $x_* \in C_\nu = C_{\Lambda_\nu}$, we have

$$\frac{\theta(b_n x + a_n)}{\theta(b_n x_* + a_n)} \to \frac{\nu(x)}{\nu(x_*)}, \tag{4.16}$$

for each $x \in C_\nu = C_{\Lambda_\nu}$.

Conversely, if the above conditions hold with a sequence of norming and centering constants $b_n > 0$ and $a_n \in \mathbb{R}$ satisfying (4.3), (4.15) and (4.8) for each $x \in \mathbb{R}$ then there exists a sequence k_n satisfying (1.3c),

$$k_n := \left\lfloor \frac{\nu(x_*) e^{-x_*}}{1 - F(b_n x_* + a_n)} \right\rfloor \tag{4.17}$$

for example, such that (4.1) holds with $G = \Lambda_\nu$, and so $F \in \mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$.

Theorem 4.3 is the counterpart of Theorem 2.2 in Grinevich [48], and if $0 \in C_\nu = C_{\Lambda_\nu}$, when it is convenient to choose $x_* = 0$ in condition (4.16), the statements of the two theorems are mathematically equivalent. It should be noted at this point that there is a significant difference between the Russian and the English version of Grinevich [48]: in the sufficiency part of Theorem 2.2 in the Russian version there is a problem similar to the one that was pointed out after the statement of Theorem 4.2 and hence the sufficiency part of Theorem 2.2 fails in the Russian version, while the English version is essentially correct. However, even in the English version there is a small gap in the proof which is filled here.

In a recent paper Canto e Castro, de Haan, and Graça Temido [15] gave characterizations for max-domains of geometric partial attraction of max-semistable laws with entirely different methods. Although the descriptions there are of a somewhat simpler form, the ones given by Theorems 4.1–4.3 above appear to be more applicable to obtain the results in Section 4.3.

Proofs

The proof of Theorem 4.1 requires the following elementary lemma.

Lemma 4.1. *Let $\{B_n\}$, $B_n > 0$ and $\{A_n\}$ be a pair of sequences satisfying*

$$\frac{B_{n+1}}{B_n} \to \gamma > 1 \qquad \text{and} \qquad \frac{A_{n+1} - A_n}{B_n} \to 0.$$

Then $A_n/B_n \to 0$. ■

Proof of Theorem 4.1. The necessity of (4.5) and (4.8) was pointed out in Section 1, hence the conditions of Lemma 4.1 are satisfied and (4.9) follows. The fact that $\operatorname{rext} F = \infty$ is now an easy consequence of (4.8), (4.5) and (4.9). Choose an arbitrary distribution function F_0 such that $1 - F_0(\cdot)$ is regularly varying at ∞ with index $-\alpha$. The statement in (4.1) can be reformulated as

$$k_n\big(1 - F(b_n x + a_n)\big) \to x^{-\alpha}\nu(\log x), \tag{4.18}$$

for each $x \in C_{\nu(\log(\cdot))}$, which implies

$$\frac{x^{\alpha}(1 - F_0(b_n x + a_n))}{x_*^{\alpha}(1 - F_0(b_n x_* + a_n))} \cdot \frac{\theta(b_n x + a_n)}{\theta(b_n x_* + a_n)} \to \frac{\nu(\log x)}{\nu(\log x_*)},$$

for all such x. Here the first factor converges to 1 owing to (4.9) and the regular variation of $1 - F_0(\cdot)$. Thus the necessity of (4.10) follows. The last statement in the necessity part of the theorem is a consequence of (4.9) and of the monotonicity of the functions $F^{k_n}(\cdot)$ and $\Phi_{\alpha,\nu}(\cdot)$.

Turning to sufficiency, we have to show that the sequence defined at (4.11) is an admissible choice indeed. Using (4.9), (4.5) and the regular variation of $1 - F_0(\cdot)$, it follows by a direct calculation that (4.18) is satisfied, yielding (4.1). It remains to show that (1.3c) holds. To this aim, observe first that (4.10) remains true if the fixed x on its left-hand side is replaced by a sequence $x_n \to x \in C_{\nu(\log(\cdot))}$, since if $x_- < x < x_+$, $x_-, x_+ \in C_{\nu(\log(\cdot))}$, then $\theta(b_n x_n + a_n)$ can be estimated as

$$\frac{1 - F(b_n x_+ + a_n)}{1 - F_0(b_n x_- + a_n)} \leq \theta(b_n x_n + a_n) \leq \frac{1 - F(b_n x_- + a_n)}{1 - F_0(b_n x_+ + a_n)}$$

for all n large enough, and hence, using (4.10) and the regular variation of $1 - F_0(\cdot)$ together with (4.8) and (4.9), it follows that

$$\begin{aligned} \frac{\nu(\log(x_+))}{\nu(\log(x_*))}\left(\frac{x_+}{x_-}\right)^{-\alpha} &= \lim_{n\to\infty} \frac{\theta(b_n x_+ + a_n)}{\theta(b_n x_* + a_n)} \cdot \frac{1 - F_0(b_n x_+ + a_n)}{1 - F_0(b_n x_- + a_n)} \leq \frac{\theta(b_n x_n + a_n)}{\theta(b_n x_* + a_n)} \\ &\leq \lim_{n\to\infty} \frac{\theta(b_n x_- + a_n)}{\theta(b_n x_* + a_n)} \cdot \frac{1 - F_0(b_n x_- + a_n)}{1 - F_0(b_n x_+ + a_n)} = \frac{\nu(\log(x_-))}{\nu(\log(x_*))}\left(\frac{x_-}{x_+}\right)^{-\alpha}, \end{aligned} \tag{4.19}$$

and the upper and lower bounds can be arbitrarily close to $\frac{\nu(\log(x))}{\nu(\log(x_*))}$. Introduce

$$\gamma_n := \frac{b_{n+1}}{b_n} x_* + \frac{a_{n+1} - a_n}{b_n} \to \gamma x_*.$$

Now,

$$\begin{aligned} \frac{k_n}{k_{n+1}} &\sim \frac{1 - F(b_{n+1} x_* + a_{n+1})}{1 - F(b_n x_* + a_n)} \\ &= \frac{1 - F_0(b_{n+1} x_* + a_{n+1})}{1 - F_0(b_n x_* + a_n)} \cdot \frac{\theta(b_n \gamma_n + a_n)}{\theta(b_n x_* + a_n)} \\ &\to \gamma^{-\alpha} \frac{\nu(\log(\gamma x_*))}{\nu(\log(x_*))} = \gamma^{-\alpha} = \frac{1}{c}. \end{aligned}$$

Here the first factor in the second row converges to $\gamma^{-\alpha}$ by (4.5), (4.9) and the regular variation of $1 - F_0(\cdot)$, while the convergence of the second factor follows using the inequalities in (4.19). Notice also that the periodicity of ν implies $\nu(\log(\gamma x_*)) = \nu(\log(x_*))$ and $\gamma x_* \in C_{\nu(\log(\cdot))}$. ∎

To prove Theorem 4.3 we need the following lemma, which is an immediate consequence of Lemma 1.3 in Resnick [68].

Lemma 4.2. *Let $F_\sharp$ be a von Mises function with auxiliary function $f(\cdot)$, $x_n \to x \in \mathbb{R}$ and $a_n \uparrow \operatorname{rext} F_\sharp$. Then*

$$\int_{a_n}^{f(a_n)x_n + a_n} \frac{1}{f(u)}\, du \to x.$$

∎

Proof of Theorem 4.3. If $F \in \mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$ then, as stated in Grinevich [46] and Grinevich [47] and quoted in the Introduction, the centering and norming constants figuring in (4.1) necessarily satisfy (4.3). The relation in (4.8) is also satisfied for *all* $x \in \mathbb{R}$, since $0 < \Lambda_\nu(x) < 1$ for all $x \in \mathbb{R}$. In particular, the case $x = 0$ shows $a_n \uparrow \operatorname{rext} F$. The second convergence statement in (4.3) and the fact that $b_n > 0$ together imply that the sequence a_n can be chosen strictly increasing. Hence there exists a $z_0 \in \mathbb{R}$, $z_0 < \operatorname{rext} F$ such that the following function is unambiguously defined for all $x \in (z_0, \operatorname{rext} F)$:

$$f(x) := \begin{cases} b_n, & \text{if } x = a_n \text{ for some } n \in \mathbb{N}, \\ \text{linear in between.} \end{cases}$$

The function f is clearly positive and absolutely continuous. Moreover,

$$\frac{b_{n+1} - b_n}{a_{n+1} - a_n} = \left(\frac{b_{n+1}}{b_n} - 1\right) \frac{b_n}{a_{n+1} - a_n} \longrightarrow 0,$$

as a consequence of (4.3). Hence $\lim_{u \uparrow \operatorname{rext} F} f'(u) = 0$, where $f'(\cdot)$ denotes the density of $f(\cdot)$ and $f(\cdot)$ is a von Mises auxiliary function. Let $F_\sharp(\cdot)$ be the von Mises function determined by $f(\cdot)$: since $\sum_{n=1}^{\infty} \frac{a_{n+1} - a_n}{b_n} = \infty$, $F_\sharp$ obtained this way is a distribution function indeed, and with this choice part (i) of the theorem is clearly satisfied.

Turning to part (ii), an equivalent form of (4.1) is

$$k_n(1 - F(b_n x + a_n)) \longrightarrow \exp(-x)\nu(x), \tag{4.20}$$

for each $x \in C_\nu = C_{\Lambda_\nu}$. Using $b_n = f(a_n)$, for the distribution function F we have

$$\frac{1 - F(b_n x + a_n)}{1 - F(b_n x_* + a_n)} = \frac{\theta(b_n x + a_n)}{\theta(b_n x_* + a_n)} \exp\left(-\int_{f(a_n)x_* + a_n}^{f(a_n)x + a_n} \frac{1}{f(u)}\, du\right).$$

By a special case of Lemma 4.2, the second factor converges to $e^{(x_* - x)}$. Since it follows from (4.20) that

$$\frac{1 - F(b_n x + a_n)}{1 - F(b_n x_* + a_n)} \longrightarrow e^{(x_* - x)} \frac{\nu(x)}{\nu(x_*)}, \qquad x \in C_\nu,$$

all this together imply part (ii) of the theorem, and thus necessity is proved.

Turning to sufficiency, we have to show that the subsequence $\{k_n\}$ defined in (4.17) satisfies (4.20) and (1.3c): as for (4.20), this can be done by a direct computation. To prove (1.3c), introduce

$$\beta_n := \frac{a_{n+1} - a_n}{b_n} + \left(\frac{b_{n+1}}{b_n} - 1\right) x_* \longrightarrow \beta,$$

where the convergence takes place by (4.3). Now,

$$\begin{aligned}
\frac{k_n}{k_{n+1}} &\sim \frac{1-F(b_{n+1}x_*+a_{n+1})}{1-F(b_nx_*+a_n)} \\
&= \frac{\theta(b_{n+1}x_*+a_{n+1})}{\theta(b_nx_*+a_n)}\exp\left\{-\int_{b_nx_*+a_n}^{b_{n+1}x_*+a_{n+1}}\frac{1}{f(u)}\,du\right\} \\
&= \frac{\theta(b_n(x_*+\beta_n)+a_n)}{\theta(b_nx_*+a_n)}\exp\left\{-\int_{f(a_n)x_*+a_n}^{f(a_n)(x_*+\beta_n)+a_n}\frac{1}{f(u)}\,du\right\} \\
&\to \frac{e^{-(x_*+\beta)}\nu(x_*+\beta)}{e^{-x_*}\nu(x_*)} = e^{-\beta} = \frac{1}{c},
\end{aligned}$$

as it should be. The convergence statement holds by part(ii) of the theorem, using also Lemma 4.2. At this point notice similarly as in the proof of Theorem 4.1 that (4.16) remains true if the fixed x on its left-hand side is replaced by a sequence $x_n \to x \in C_\nu$ and that $\nu(x_*) = \nu(x_*+\beta)$ and $x_*+\beta \in C_\nu$ by the periodicity of ν. ∎

4.3. The Structure of Max-Domains of Geometric Partial Attraction: The Merge

Before starting any investigation into the structure of max-domains of geometric partial attraction observe that none of these domains is void: $\Lambda_\nu \in \mathbb{MD}_{\text{gp}}(\Lambda_\nu)$, $\Phi_{\alpha,\nu} \in \mathbb{MD}_{\text{gp}}(\Phi_{\alpha,\nu})$ and $\Psi_{\alpha,\nu} \in \mathbb{MD}_{\text{gp}}(\Psi_{\alpha,\nu})$. Hence the structure we want to deal with is not trivial, but before tackling with deeper problems we need some notation. Introduce for $n \in \mathbb{N}$

$$n^* := \min\{m \in \mathbb{N} : k_m \geq n\}.$$

If the subsequence k_n satisfies (1.3c) with $c > 1$, which is assumed throughout with the exception of Theorem 4.7 below, then, for all n large enough, k_{n^*} is nothing but the smallest element of $\{k_n\}$ not smaller than n. We shall also need the sequence

$$\gamma_n := \frac{n}{k_{n^*}}.$$

Clearly, with the notation from Theorem 2.6,

$$\gamma_n = \frac{1}{\gamma(1/n)}.$$

Obviously, $\liminf \gamma_n = \frac{1}{c}$ and $\limsup \gamma_n = 1$. The first theorem below, the analogue of the Theorem 2.6, generalizes Theorem 2.7.

Theorem 4.4. *(Merge Theorem for Max-Semistable Laws) Assume that $F \in \mathbb{MD}_{\mathrm{gp}}(G)$ with centering and norming sequences $\{a_n\}$ and $\{b_n\}$, where G is an arbitrary max-semistable law. Then*

$$\mathcal{L}\big(F^n(b_n^\circ x + a_n^\circ),\, G^{\gamma_n}(x)\big) \to 0, \tag{4.21}$$

where the sequences $\{a_n^\circ\}_{n=1}^\infty$ and $\{b_n^\circ\}_{n=1}^\infty$ are defined as

$$a_n^\circ = a_{n^*} \quad \text{and} \quad b_n^\circ = b_{n^*}.$$

Furthermore, if the distribution function $G(\cdot)$ is continuous then (4.21) may be strengthened to

$$\sup_{\infty < x < \infty} |F^n(b_n^\circ x + a_n^\circ) - G^{\gamma_n}(x)| \to 0. \tag{4.22}$$

Of course, G is given by one of the expressions at (4.4), (4.6) or (4.7) and G is continuous if and only if the function $\nu(\cdot)$ figuring in the corresponding expression is continuous. If G equals Λ_ν, $\Phi_{\alpha,\nu}$ or $\Psi_{\alpha,\nu}$, then the distribution functions G^{γ_n} can be written as $\Lambda_{\gamma_n\nu}$, $\Phi_{\alpha,\gamma_n\nu}$ or $\Psi_{\alpha,\gamma_n\nu}$, respectively, and they are all max-semistable of course. In the next corollary the centering and norming constants are changed and hence the set of the limiting distributions is different. The idea is to achieve that the limiting distributions parametrized by $\gamma_n = \frac{1}{c}$ and $\gamma_n = 1$ be exactly the same rather than only of the same type.

Corollary 4.1. *(i) Assume $F \in \mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$ with centering and norming sequences $\{a_n\}$ and $\{b_n\}$. Introduce*

$$a_n^{(1)} := a_{n^*} + \frac{\log \gamma_n}{\beta}(a_{n^*} - a_{n^*-1}) \quad \text{and} \quad b_n^{(1)} := f(a_n^{(1)}),$$

where $f(\cdot)$ is the von Mises auxiliary function from Theorem 4.3. Then

$$\mathcal{L}\left(F^n(b_n^{(1)}x + a_n^{(1)}),\, \exp\big(-\exp(-x)\nu(x + \log \gamma_n)\big)\right) \to 0.$$

(ii) Assume $F \in \mathbb{MD}_{\mathrm{gp}}(\Phi_{\alpha,\nu})$ with centering and norming sequences $\{a_n\}$ and $\{b_n\}$. Introduce

$$a_n^{(2)} := 0 \quad \text{and} \quad b_n^{(2)} := \gamma_n^{1/\alpha} b_{n^*}.$$

Then

$$\mathcal{L}\left(F^n(b_n^{(2)}x + a_n^{(2)}),\, \exp\left(-x^{-\alpha}\nu\big(\log(\gamma_n^{1/\alpha}x)\big)\right)\right) \to 0.$$

(iii) Assume $F \in \mathbb{MD}_{\mathrm{gp}}(\Psi_{\alpha,\nu})$ with centering and norming sequences $\{a_n\}$ and $\{b_n\}$. Introduce

$$a_n^{(3)} := \operatorname{rext} F \quad \text{and} \quad b_n^{(3)} := \gamma_n^{-1/\alpha} b_{n^*}.$$

Then

$$\mathcal{L}\Big(F^n(b_n^{(3)}x + a_n^{(3)}),\ \exp\big(-|x|^\alpha \nu(\log|\gamma_n^{-1/\alpha}x|)\big)\Big) \to 0.$$

■

Of course, if ν is continuous in any of the three statements then a uniform variant of that statement is true, as in (4.22). The sets of the attracting distributions are now $\Lambda_{\gamma_n\nu}(x+\log\gamma_n)$, $x \in \mathbb{R}$, $\Phi_{\alpha,\gamma_n\nu}(\gamma_n^{1/\alpha}x)$, $x > 0$, and $\Psi_{\alpha,\gamma_n\nu}(\gamma_n^{-1/\alpha}x)$, $x < 0$, respectively. The corollary follows from the Merge Theorem and Theorems 4.1–4.3 by a straightforward calculation, so its proof is omitted. We notice that using Satz 4.5 in Becker-Kern [1], Theorem 4.4 and Corollary 4.1 can be extended to not necessarily identically distributed cases, but we will not go into this.

There is another consequence of Theorem 4.4 which is worth mentioning. A distribution function F is called *max-stochastically compact* if there exist centering and norming sequences $\{a_n\}$ and $\{b_n\}$ such that for every subsequence $\{n(m)\}_{m=1}^\infty$ there exists a further subsequence $\{n(m(r))\}_{r=1}^\infty \subset \{n(m)\}_{m=1}^\infty$ such that

$$\Big(F\big(b_{n(m(r))}x + a_{n(m(r))}\big)\Big)^{n(m(r))} \to K(x), \qquad \text{as } r \to \infty,$$

for each $x \in C_K$, where $K(\cdot)$ is a non-degenerate distribution function depending generally on $\{n(m(r))\}$. An easy consequence of the Merge Theorem is that each $F \in \mathbb{MD}_{\mathrm{gp}}(G)$ with any max-semistable G is max-stochastically compact. One should expect that max-stochastic compactness follows easily from a combination of the representations in Theorems 4.1–4.3 and one of the necessary and sufficient conditions for max-stochastic compactness in de Haan and Resnick [51] — Theorem 3 there seems to be particularly applicable for this purpose. That this is not the case is due to the fact that the functions $\theta(\cdot)$ in Theorems 4.1–4.3 need not be bounded. Also, this is the reason that it is unclear whether a statement similar to Corollary 2.4 on the moments on the attracted distributions holds. The verification of max-stochastic compactness however, which implies certain compactness properties for intermediate sums by Csörgő and Mason [31], too, will be the first step of the proof of Theorem 4.4. Theorem 4.6 below (an analogue of Theorem 2.5) states much more than max-stochastic compactness: it exactly determines the subsequences along which convergence takes place. However, before proceeding in this direction we need a preliminary result.

Theorem 4.5. *For the non-constant periodic function $\nu(\cdot)$ introduce*

$$T_\nu := \min\{t > 0 : \nu(x+t) = \nu(x),\ x \in \mathbb{R}\},$$

its smallest positive period.

(i) Suppose $F \in \mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$ along a subsequence k_n satisfying (1.3c) with $c > 1$ and let $\nu(\cdot)$ be non-constant. Then there exists a positive integer N such that $\beta = NT_\nu$ with β

from (4.2) and there exists a subsequence $\{\mathbf{k}_n\}_{n=1}^{\infty}$ *and centering and norming sequences* $\{\mathbf{a}_n\}_{n=1}^{\infty}$, $\{\mathbf{b}_n\}_{n=1}^{\infty}$ *such that*

$$\frac{\mathbf{k}_{n+1}}{\mathbf{k}_n} \to \mathbf{c} := c^{1/N} \tag{4.23}$$

and

$$\Big(F\big(\mathbf{b}_n x + \mathbf{a}_n\big)\Big)^{\mathbf{k}_n} \to G(x), \qquad x \in C_G, \tag{4.24}$$

with $G = \Lambda_\nu$ *jointly hold.*

(ii) Suppose $F \in \mathbb{MD}_{\mathrm{gp}}(\Phi_{\alpha,\nu})$ *(or* $F \in \mathbb{MD}_{\mathrm{gp}}(\Psi_{\alpha,\nu})$*) along a subsequence* k_n *satisfying (1.3c) with* $c > 1$ *and let* $\nu(\cdot)$ *be non-constant. Then there exists a positive integer* N *such that* $|\log\gamma| = NT_\nu$ *and there exists a subsequence* $\{\mathbf{k}_n\}_{n=1}^{\infty}$ *and centering and norming sequences* $\{\mathbf{a}_n\}_{n=1}^{\infty}$, $\{\mathbf{b}_n\}_{n=1}^{\infty}$ *such that (4.23) and (4.24) with* $G = \Phi_{\alpha,\nu}$ *(or with* $G = \Psi_{\alpha,\nu}$*) are satisfied.*

Observe that $T_\nu = \frac{\log c}{\alpha}$ where $\alpha = 1$ if $F \in \mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$. We may and do assume now without loss of generality that the sequence $\{\gamma_n\}$ is based on a subsequence $\{k_n\}$ such that $k_{n+1}/k_n \to \mathbf{c}$.

Theorem 4.6. *As before, let* $F \in \mathbb{MD}_{\mathrm{gp}}(G)$ *for some max-semistable but not max-stable law* G. *Suppose that for some subsequence* $\{n(r)\}_{r=1}^{\infty} \subset \mathbb{N}$ *and centering and norming sequences* $\{A_{n(r)}\}$ *and* $\{B_{n(r)}\}$

$$\Big(F\big(B_{n(r)}x + A_{n(r)}\big)\Big)^{n(r)} \to K(x), \qquad \text{as } r \to \infty, \tag{4.25}$$

for each $x \in C_K$, *where* $K(\cdot)$ *is a non-degenerate distribution function. Then necessarily the sequence* $\{\gamma_{n(r)}\}_{r=1}^{\infty}$ *either converges to some limit* $\kappa \in [1/\mathbf{c}, 1]$, *or has exactly two limit points, 1 and* $1/\mathbf{c}$ *(in which case we define* $\kappa = 1$*). Introducing the centering and norming sequences as in Corollary 4.1, if* $F \in \mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$ *then* $K(x) = \Lambda_{\kappa\nu}(\delta x + \log\kappa + \mu)$, *where*

$$\delta = \lim_{r\to\infty} \frac{B_{n(r)}}{b^{(1)}_{n(r)}} \quad \text{and} \quad \mu = \lim_{r\to\infty} \frac{A_{n(r)} - a^{(1)}_{n(r)}}{b^{(1)}_{n(r)}}.$$

If $F \in \mathbb{MD}_{\mathrm{gp}}(\Phi_{\alpha,\nu})$ *then* $K(x) = \Phi_{\alpha,\kappa\nu}(\delta\kappa^{1/\alpha}x + \mu)$, *where*

$$\delta = \lim_{r\to\infty} \frac{B_{n(r)}}{b^{(2)}_{n(r)}} \quad \text{and} \quad \mu = \lim_{r\to\infty} \frac{A_{n(r)}}{b^{(2)}_{n(r)}}.$$

If $F \in \mathbb{MD}_{\mathrm{gp}}(\Psi_{\alpha,\nu})$ *then* $K(x) = \Psi_{\alpha,\kappa\nu}(\delta\kappa^{-1/\alpha}x + \mu)$, *where*

$$\delta = \lim_{r\to\infty} \frac{B_{n(r)}}{b^{(3)}_{n(r)}} \quad \text{and} \quad \mu = \lim_{r\to\infty} \frac{A_{n(r)} - \operatorname{rext} F}{b^{(3)}_{n(r)}}.$$

Conversely, if for a subsequence $\{n(r)\}_{r=1}^{\infty}$ *the sequence* $\{\gamma_{n(r)}\}_{r=1}^{\infty}$ *converges to some limit* $\kappa \in [1/c, 1]$, *or has exactly two limit points,* 1 *and* $1/c$ *(in which case we put* $\kappa = 1$*), then (4.25) holds with a limiting distribution* $\Lambda_{\kappa\nu}$, $\Phi_{\alpha,\kappa\nu}$ *or* $\Psi_{\alpha,\kappa\nu}$ *if* $F \in \mathbb{MD}_{\rm gp}(\Lambda_\nu)$, $F \in \mathbb{MD}_{\rm gp}(\Phi_{\alpha,\nu})$ *or* $F \in \mathbb{MD}_{\rm gp}(\Psi_{\alpha,\nu})$, *respectively, provided the centering and norming sequences are chosen as in Corollary 4.1.*

For any max-semistable law G consider the set consisting of all distribution functions of the form $G^\kappa(bx + a)$, $x \in \mathbb{R}$, where $\kappa, b > 0$ and $a \in \mathbb{R}$. This set is called the *family* of G, its elements are all max-semistable and $F \in \mathbb{MD}_{\rm gp}(G)$ implies that $F \in \mathbb{MD}_{\rm gp}(G')$ for all G' from the family of G. Theorem 4.6 entails that all the non-degenerate subsequential weak limits of maxima of independent random variables with distribution function $F \in \mathbb{MD}_{\rm gp}(G)$ are in the family of G. The limiting distributions in Theorem 4.4 and Corollary 4.1 are chosen of course from the family of G. It also follows that if $\mathbb{MD}_{\rm gp}(G_1) \cap \mathbb{MD}_{\rm gp}(G_2) \neq \emptyset$ for two max-semistable laws G_1 and G_2 then $\mathbb{MD}_{\rm gp}(G_1) = \mathbb{MD}_{\rm gp}(G_2)$. Another consequence of Theorem 4.6 is the following Corollary 4.2, which is an analogue of Corollary 2.5.

Corollary 4.2. *Suppose that (4.1) holds for a max-semistable but not max-stable limiting law* G *along a subsequence* $\{k_n\}$ *for which* $k_{n+1}/k_n \to c > 1$. *If*

$$\mathbb{P}\big(b_n'^{-1}(M_{k_n'} - a_n') \le x\big) \to K(x), \qquad x \in C_K,$$

for a non-degenerate limiting distribution K *with a subsequence* $\{k_n'\}$ *such that* $\lim_{n\to\infty} k_{n+1}'/k_n' = \lim_{n\to\infty} k_{n+1}/k_n = c$, *then* $\lim_{n\to\infty} k_n'/k_n = \lambda$, $\lim_{n\to\infty} b_n'/b_n = b$ *and* $\lim_{n\to\infty}(a_n' - a_n)/b_n \to a$ *for some constants* $a \in \mathbb{R}$ *and* $\lambda, b \in (0, \infty)$.

We mentioned already that if $c = 1$ in (1.3c) then the limiting distribution is necessarily max-stable. We stipulated $c > 1$ only to avoid degenerate cases and thus to simplify the formulation of our results. Similarly, the condition that $\nu(\cdot)$ is non-constant is nothing but requiring that the underlying distribution is max-semistable but not max-stable. The last theorem in this section shows how max-stable laws are inserted within the framework of the theory of max-semistable laws.

Theorem 4.7. *Suppose (4.1) holds along a subsequence* $\{k_n\}_{n=1}^{\infty}$ *satisfying (1.3c) with* $c = 1$. *Then the limiting distribution* G *figuring in (4.1) is necessarily max-stable. Furthermore, if (4.1) holds with a max-stable limiting law along a subsequence* $\{k_n\}$ *satisfying (1.3c) with arbitrary* $c \ge 1$ *then there exist sequences of centering and norming constants such that (4.1) holds with* $k_n = n$. *Hence the max-domain of geometric partial attraction and the max-domain of attraction of max-stable laws coincide.*

In the definition of semistable law (in the sense of sums) it was enough to assume that instead of (1.3c) the subsequence $\{k_n\}$ satisfies the weaker condition (1.3a): the same is true for max-semistable laws. If we have convergence along a subsequence satisfying (1.3a) then the limiting distribution is automatically max-semistable. Furthermore, if for some F there exist centering and norming sequences $\{a_n\}$ and $\{b_n\}$ such that $b_n^{-1}(M_{k_n} - a_n)$ converges to some (necessarily max-semistable) G along a subsequence satisfying (1.3b) then $F \in \mathbb{MD}_{\mathrm{gp}}(G)$, hence conditions (1.3b) and (1.3c) are equivalent even with respect to the determination of the attracted distributions. However, the set of distributions attracted to G in the sense of maxima along subsequences satisfying (1.3a) is essentially wider than $\mathbb{MD}_{\mathrm{gp}}(G)$. The proof of these statements is parallel to the corresponding ones in Section 2.

Proofs

Proof of Theorem 4.4. For the first statement of the theorem it suffices to show that any subsequence $\{n(m)\}_{m=1}^{\infty} \in \mathbb{N}$ contains a further subsequence $\{n(m(r))\}_{r=1}^{\infty} \subset \{n(m)\}_{m=1}^{\infty}$ such that (4.21) holds along $\{n(m(r))\}$. Indeed, any subsequence $\{n(m)\}$ contains a further subsequence $\{n(m(r))\}$ such that $\gamma_{n(m(r))}$ converges to some limit point $\kappa \in [\frac{1}{c}, 1]$. Now,

$$F^{n(m(r))}(b^{\circ}_{n(m(r))}x + a^{\circ}_{n(m(r))}) = \left(F^{k_{n(m(r))^*}}\left(b_{n(m(r))^*}x + a_{n(m(r))^*}\right)\right)^{\gamma_{n(m(r))}}. \quad (4.26)$$

Here $F^{k_{n(m(r))^*}}\left(b_{n(m(r))^*}x + a_{n(m(r))^*}\right) \to G(x)$ for all $x \in C_G$ by $F \in \mathbb{MD}_{\mathrm{gp}}(G)$. However, $C_G = C_{G^\kappa}$ in all the three possible max-semistable cases, and so the expression in (4.26) converges to $G^\kappa(x)$ for each $x \in C_{G^\kappa}$. In other words

$$\mathcal{L}(F^{n(m(r))}(b^{\circ}_{n(m(r))}x + a^{\circ}_{n(m(r))}), G^\kappa(x)) \to 0, \qquad \text{as } r \to \infty. \quad (4.27)$$

Observe that $G^{\kappa_n}(x) \to G^\kappa(x)$ for all sequences $\{\kappa_n\}_{n=1}^{\infty}$ such that $\kappa_n \to \kappa > 0$ and for all $x \in \mathbb{R}$. This evidently implies

$$\mathcal{L}(G^\kappa, G^{\gamma_{n(m(r))}}) \to 0, \qquad \text{as } r \to \infty. \quad (4.28)$$

Now (4.21) follows using the triangle inequality for the Lévy distance. The statement in (4.22) can be proved analogously, using this time the uniform versions of (4.27) and (4.28). ■

Proof of Theorem 4.5. First we deal with part (i). Since $\beta > 0$ and T_ν are both periods of the non-constant function $\nu(\cdot)$ and T_ν is its smallest period, necessarily there exists $N \in \mathbb{N}$ such that $\beta = NT_\nu$.

Introduce $\mathbf{c} := c^{1/N}$ and for $j \in \{1, \dots, N\}$ define the following sequences:

$$k_j(n) := \lfloor \mathbf{c}^{1-j} k_n \rfloor, \qquad a_j(n) := a^{(1)}_{k_j(n)}, \qquad b_j(n) := b^{(1)}_{k_j(n)},$$

where $\{a^{(1)}_n\}$ and $\{b^{(1)}_n\}$ are from Corollary 4.1. Since $\gamma_{k_j(n)} \to \mathbf{c}^{1-j}$,

$$F^{k_j(n)}(b_j(n)x + a_j(n)) \to \exp\left(-\exp(-x)\nu(x + \log \mathbf{c}^{1-j})\right)$$

for all $x \in \mathbb{R}$ such that $x + \log \mathbf{c}^{1-j}$ is a continuity point of ν and for each $j \in \{1, \ldots, N\}$. Recalling from Section 1 that $\beta = \log c$, we get $\log \mathbf{c}^{1-j} = (1-j)T_\nu$. The periodicity of $\nu(\cdot)$ implies that $x + \log \mathbf{c}^{1-j}$ is a continuity point of $\nu(\cdot)$ if and only if $x \in C_\nu$ and that $\nu(x + \log \mathbf{c}^{1-j}) = \nu(x)$. Hence

$$F^{k_j(n)}(b_j(n)x + a_j(n)) \to \exp\big(-\exp(-x)\nu(x)\big) = \Lambda_\nu(x) \qquad \text{for all } x \in C_\nu = C_{\Lambda_\nu},$$

for each $j \in \{1, \ldots, N\}$. In other words, weak convergence takes place along any of the subsequences $\{k_j(n)\}_{n=1}^\infty$. Hence we also have convergence along the eventually strictly increasing union sequence $\{\mathbf{k}_n\}_{n=1}^\infty := \bigcup_{j=1}^N \{k_j(n)\}_{n=1}^\infty$. Obviously (4.23) is satisfied and (4.24) holds as well if the centering and norming constants $\mathbf{a}_n$ and $\mathbf{b}_n$ are chosen from the sets $\bigcup_{j=1}^N \{a_j(n)\}_{n=1}^\infty$ and $\bigcup_{j=1}^N \{b_j(n)\}_{n=1}^\infty$ in the natural way.

The proof of part (ii) is entirely similar, this time using part (ii) or part (iii) of Corollary 4.1, respectively. ■

Proof of Theorem 4.6. The sufficiency part of the theorem is an easy consequence of Corollary 4.1. To prove necessity it suffices to show that for any max-semistable distribution function G

$$G^{\lambda_1} \text{ and } G^{\lambda_2} \text{ are of different type for } 1/\mathbf{c} < \lambda_1 < \lambda_2 \le 1. \tag{4.29}$$

Once (4.29) is demonstrated then all the statements in the necessity part follow by applying the Merge Theorem, Corollary 4.1 and the Convergence of Types Theorem.

We may assume without loss of generality that $\lambda_2 = 1$ in (4.29) and so we simply write λ instead of λ_1. First we prove (4.29) for $G = \Lambda_\nu$. Assume to the contrary that there exist $b > 0$, $a \in \mathbb{R}$ and $\lambda \in (1/\mathbf{c}, 1)$ such that

$$\exp\big[-\exp(-x)\nu(x)\big] = \exp\big[-\exp(-bx - a)\lambda\nu(bx + a)\big], \qquad x \in \mathbb{R}.$$

Taking logarithm, this yields

$$\frac{\lambda^{-1}\exp(-x)}{\exp(-bx - a)} = \frac{\nu(bx + a)}{\nu(x)}, \qquad x \in \mathbb{R}. \tag{4.30}$$

Since the positive function $\nu(\cdot)$ is bounded away both from 0 and infinity, (4.30) forces $b = 1$. However, if $b = 1$ then the left-hand side of (4.30) is independent of x, hence constant, which may only happen if $\nu(x) = \nu(x + a)$ for all $x \in \mathbb{R}$. Then necessarily $a = mT_\nu$ for some $m \in \mathbb{Z} = \{\ldots -1, 0, 1, \ldots\}$ with T_ν from the preceding theorem and $\lambda = \exp(mT_\nu)$. Since $\log \mathbf{c} = T_\nu$ this is incompatible with $\lambda \in (1/\mathbf{c}, 1)$.

The proof of (4.29) is indirect in case of $G = \Phi_{\alpha,\nu}$ as well. Assume to the contrary that there exist $b > 0$, $a \in \mathbb{R}$ and $\lambda \in (1/\mathbf{c}, 1)$ such that

$$\exp\big(-x^{-\alpha}\nu(\log x)\big) = \exp\big(-(bx + a)^{-\alpha}\lambda\nu(\log(bx + a))\big), \qquad x > 0.$$

Letting $x \downarrow 0$ shows that $a = 0$. This yields

$$\lambda^{-1} b^{\alpha} = \frac{\nu(\log x + \log b)}{\nu(\log x)}, \qquad x > 0. \tag{4.31}$$

The left-hand side of (4.31) is independent of x just as above, which may only happen if $\log b = mT_\nu$ for some $m \in \mathbb{Z}$ and $\lambda = \exp(\alpha m T_\nu)$ which contradicts $\lambda \in (1/\mathbf{c}, 1)$ by $\alpha T_\nu = \log \mathbf{c}$. The case $G = \Psi_{\alpha,\nu}$ being entirely similar, the theorem is completely proved. ∎

Proof of Corollary 4.2. Using the notation from Theorem 4.5 and 4.6, let γ_n be defined by $\{\mathbf{k}_n\}$: then $\{\gamma_{k'_n}\}$ either converges to some limit $\kappa' \in [1/\mathbf{c}, 1]$, or has exactly two limit points, 1 and $1/\mathbf{c}$ (in which case we define $\kappa' = 1$). The same is true for γ_{k_n}, but if γ_{k_n} converges then the limit point κ is necessarily 1 or $1/\mathbf{c}$. In either case, since $\lim_{n\to\infty} k'_{n+1}/k'_n = \lim_{n\to\infty} k_{n+1}/k_n$, we conclude that $\lim_{n\to\infty} k'_n/k_n = \lambda$ with $\lambda = \kappa' \mathbf{c}^M$, for some $M \in \mathbb{N}$. Now necessarily $K(x) = G^{\lambda}(bx + a)$ and the remaining statements follow from the Convergence of Types Theorem. ∎

Proof of Theorem 4.7. Suppose (4.1) holds along a subsequence $\{k_n\}_{n=1}^{\infty}$ satisfying (1.3c) with $c = 1$. Then for any given $c' > 1$ we may choose a subsequence $\{k_{n(r)}\}_{r=1}^{\infty}$ such that $k_{n(r+1)}/k_{n(r)} \to c'$, as $r \to \infty$. Using the theory developed so far this means that the limiting distribution G belongs to one of the max-semistable classes Λ_ν, $\Phi_{\alpha,\nu}$ or $\Psi_{\alpha,\nu}$, and that the function ν is periodic with period $\frac{\log c'}{\alpha}$ where $\alpha = 1$ in case of Λ_ν. Since c' was arbitrary the function ν is necessarily constant, thus the limiting law G is max-stable and belongs to one of the types $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, $\Phi_\alpha(x) = \exp(-x^{-\alpha})$, $x > 0$, and $\Psi_\alpha(x) = \exp(-|x|^{\alpha})$, $x < 0$. The remaining statement of the theorem is now a consequence of Corollary 4.1, which can be applied since the subsequence k_n satisfies (1.3c) with $c > 1$. Clearly, if $\nu(\cdot)$ is constant then the family of the attracting distributions in Corollary 4.1 is identically G and weak convergence takes place along $\{n\}$ as claimed. ∎

5. Almost sure limit theorems

5.1. Introduction

Many authors investigated the connection between the weak limit theorem

$$\mathbb{P}\left\{\frac{S_n - B_n}{A_n} \le x\right\} \to G(x), \qquad \text{for any } x \in C_G, \tag{5.1}$$

and the corresponding pointwise result

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\, I\left\{\frac{S_r - B_r}{A_r} \le x\right\} \to G(x), \qquad \text{a.s. for any } x \in C_G, \tag{5.2}$$

where $S_n = X_1 + \cdots + X_n$, $A_n > 0$ and B_n are some numerical sequences and $I(\cdot)$ denotes the indicator function.

Berkes and Dehling [11] proved the surprising result that under mild moment conditions on the X_n, the relation in (5.1) implies the one in (5.2), despite the almost sure nature of the latter. If the random variables $X_1, X_2, \ldots$ are identically distributed, then the limiting distribution in (5.1) is necessarily stable with some exponent $\alpha \in (0, 2\,]$, and the theorem of Berkes and Dehling yields the well-known results of Brosamler [14], Schatte [69], Lacey and Philipp [56] and Fisher [41] in the case $EX_1^2 < \infty$, and that of Peligrad and Révész [66] in the case $0 < \alpha < 2$.

Not only sums but more general functionals of the sequence $\{X_n\}_{n=1}^{\infty}$ are also investigated in the literature, including the case of maxima. In particular, Fahrner and Stadtmüller [39], and Cheng, Peng, and Qi [17] proved that

$$\mathbb{P}(b_n^{-1}(M_n - a_n) \le x) \to G(x), \qquad x \in C_G,$$

with any non-degenerate (hence max-stable) G implies

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\, I\left\{\frac{M_r - a_r}{b_r} \le x\right\} \to G(x), \qquad \text{a.s. for any } x \in C_G.$$

For a recent survey of results in the area of "logarithmic" limit theorems of type (5.2), see Berkes [6].

In the converse direction, Berkes, Dehling and Móri [12] constructed examples for which (5.2) holds while (5.1) does not (and Móri [64] discovered interesting natural examples, different from sums, with the same feature). Berkes, Csáki and Csörgő [9] recently showed that this amusing phenomenon for sums of independent and identically distributed random variables obtains not just for esoteric examples, but also in a well-known classical situation: for the underlying distribution of the St. Petersburg game the almost sure statement in (5.2) holds with a suitable G, despite the fact that convergence in (5.1) does not take place along the whole $\{n\} = \mathbb{N}$, and the same is true for the maximal gains in a sequence of games. Our aim here is to extend these results and thus to show that the phenomenon holds not just for isolated examples, but in fact remains true for a whole class of distributions, namely, for those in the domain of geometric partial attraction of semistable laws and those in the max-domain of geometric partial attraction of max-semistable laws. In the first subsection below we state and prove our results for sums and in the second one for maxima.

5.2. Almost sure limit theorems for sums

To state the main results, recall the notation from Section 2, and introduce for $\lambda > 0$ the functions

$$\psi_j^{\lambda,\alpha}(s) := \lambda^{-1/\alpha}\psi_j^{\alpha}(s/\lambda) = -M_j(s/\lambda)s^{-1/\alpha}, \qquad s \in (0,\infty),$$

$j = 1, 2$, and the distribution function

$$G_\lambda(x) := \mathbb{P}\{V_{0,0}(\psi_1^{\lambda,\alpha}, \psi_2^{\lambda,\alpha}, 0) \le x\}, \qquad x \in \mathbb{R}.$$

Let now $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ along a subsequence $\{k_n\}$ satisfying (1.3c), in which there is no loss of generality to assume that $c > 1$. Introduce as in Section 4,

$$\gamma_n := \frac{1}{\gamma(1/n)} = \frac{n}{\min\{k_m : k_m \ge n\}}, \qquad n \in \mathbb{N}.$$

Again, it is easy to see that $\gamma_n \le 1$ and $\liminf_{n\to\infty} \gamma_n = 1/c$ for the c figuring in (1.3c). Now the merge result in Theorem 2.6 for full sums can be restated as follows:

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{\frac{\sum_{j=1}^n X_j - n\int_{\frac{1}{n}}^{1-\frac{1}{n}} Q(u)\,du}{n^{1/\alpha}\ell(1/n)} \le x\right\} - G_{\gamma_n}(x)\right| \to 0. \tag{5.3}$$

Our first theorem is now the following:

Theorem 5.1. *Assume that $F \in \mathbb{D}_{\mathrm{gp}}(G_{\psi_1^\alpha,\psi_2^\alpha,0})$ along a subsequence $\{k_n\}$ satisfying (1.3c) with some $c > 1$. Then*

$$\frac{1}{\log n}\sum_{r=1}^n \frac{1}{r} I\left\{\frac{\sum_{j=1}^r X_j - r\int_{\frac{1}{r}}^{1-\frac{1}{r}} Q(u)\,du}{r^{1/\alpha}\ell(1/r)} \le x\right\} \to \frac{1}{\log c}\int_{\frac{1}{c}}^1 \frac{G_\gamma(x)}{\gamma}\,d\gamma \tag{5.4}$$

almost surely for all $x \in \mathbb{R}$.

Next we are concerned with determining all possible limits if in Theorem 5.1 the norming factor $r^{1/\alpha}\ell(1/r)$ is replaced by an arbitrary one, a_r, say. The motivation comes from Berkes and Csáki [7], where it was shown that in the case when the random variables $X_1, X_2, \ldots$ have zero mean and finite variance then the class of all possible almost sure limiting distribution functions of the random functions

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}I\left\{\frac{\sum_{i=1}^{r}X_i}{a_r}\le x\right\}, \quad x\in\mathbb{R},$$

consists of all scale mixtures of the standard normal distribution function. Similar results were proved in [7] also for the case when the underlying distribution is attracted to an arbitrary stable law.

Before stating the corresponding result for our semistable case here, we need to introduce some notation. Given $c > 1$, a non-negative function $q(\cdot)$ on the interval $[1/c, 1]$ will be called a step-function if it is piecewise constant, i.e. there exist $\gamma_0 = 1/c < \gamma_1 < \cdots < \gamma_{k-1} < \gamma_k = 1$ and non-negative numbers $q_1, \ldots, q_{k-1}, q_k$ such that $q(\gamma) = \sum_{i=1}^{k} q_i\, I\{\gamma \in [\gamma_{i-1}, \gamma_i)\}$ for some $k \in \mathbb{N}$. Let $\mathcal{A}$ be the set of such step-functions. For $q \in \mathcal{A}$ define

$$J(x) = J(q\,; x) = \frac{1}{\log c}\int_{\frac{1}{c}}^{1}\frac{G_\gamma\left(q(\gamma)x\right)}{\gamma}\,d\gamma, \quad x\in\mathbb{R}.$$

Define the class of functions $\mathcal{J} := \{J(q\,;\cdot) : q \in \mathcal{A}\}$ and let $\mathcal{J}_*$ be the set of finite convex mixtures of $J \in \mathcal{J}$, i.e. the set of functions of the form

$$p_1J_1(\cdot)+\cdots+p_NJ_N(\cdot), \text{ where } p_1,\ldots,p_N\ge 0,\ \sum_{i=1}^{N}p_i=1,\ J_1(\cdot),\ldots,J_N(\cdot)\in\mathcal{J}$$

for some $N \in \mathbb{N}$. Finally, let $\mathcal{J}^*$ be the weak closure of $\mathcal{J}_*$.

Theorem 5.2. *Assume the condition of Theorem 5.1. Then there exists a positive numerical sequence* $\{a_n\}$ *such that*

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}I\left\{\frac{\sum_{j=1}^{r}X_j - r\int_{\frac{1}{r}}^{1-\frac{1}{r}}Q(u)\,du}{a_r}\le x\right\} = J(x), \tag{5.5}$$

almost surely for all $x \in C_J$, *if and only if* $J \in \mathcal{J}^*$.

A special case of Theorem 2.5 shows that independent variables from the domain of geometric partial attraction of a semistable distribution function G_1 also possess a certain closure or maximum property: the non-degenerate partial asymptotic laws of their sums along arbitrary subsequences all belong to the 'family' around G_1, i.e. they are of the form $\delta G_\lambda + d$, where $\delta, \lambda > 0$, $d \in \mathbb{R}$. Theorem 5.3 below, which is an analogue of Theorem 3 in [7], shows that in a sense the maximum property is preserved for almost sure distributional limits as well.

Theorem 5.3. *Assume the conditions of Theorem 5.1. If the logarithmic averages on the left-hand side of (5.5) converge to some limit $J(x)$ almost surely for each $x \in C_J$ along some subsequence $\{n_k\}_{k=1}^\infty$, then necessarily $J \in \mathcal{J}^*$. Furthermore, there exists a universal positive norming sequence $\{a_n^*\}_{n=1}^\infty$ for which the totality of subsequential almost sure limits of the corresponding logarithmic averages is identical with the whole class $\mathcal{J}^*$.*

It is natural to ask all sorts of questions concerning the limiting class $\mathcal{J}^*$. Are there more compact descriptions of it? Are all the proper distribution functions in $\mathcal{J}^*$ infinitely divisible? Does it contain normal distribution functions? Does it even contain infinitely divisible distribution functions? All these questions are open. It is easy to see that $\mathcal{J}^*$ contains improper distribution functions, and we conjecture that $\mathcal{J}^*$ contains proper distribution functions that are not infinitely divisible and it probably does not contain normal distribution functions. However, we see that all distribution functions of the form

$$J(x) = \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma\left(q(\gamma)x\right)}{\gamma}\, d\gamma, \quad x \in \mathbb{R},$$

are in $\mathcal{J}^*$ for any $q \in \mathcal{L}_1^+ = \mathcal{L}_1^+[1/c, 1]$, the class of non-negative integrable functions on $[1/c, 1]$. Hence, all distribution functions of the form

$$J(x) = \frac{1}{\log c} \int_{\mathcal{L}_1^+} \int_{\frac{1}{c}}^{1} \frac{G_\gamma\left(q(\gamma)x\right)}{\gamma}\, d\gamma\, \mu(dq), \quad x \in \mathbb{R}, \tag{5.6}$$

are also in $\mathcal{J}^*$ for any probability measure μ on $\mathcal{L}_1^+$. But it is unclear whether the whole class $\mathcal{J}^*$ is contained if in (5.6) we allow arbitrary sub-probability measures μ on $\mathcal{L}_1^+$, and attach the weight $1 - \mu(\mathcal{L}_1^+)$ to the point $x = 0$, or whether the class thus obtained is wider than $\mathcal{J}^*$.

Proofs

Lemma 5.1. *Let $\{X_n\}$ be a sequence of independent identically distributed random variables with partial sums S_n and assume that for some numerical sequences $\{\alpha_n\}$ and $\{\beta_n\}$, satisfying*

$$C_1(l/k)^\lambda \le \alpha_l/\alpha_k \le C_2(l/k)^\lambda \qquad \text{for any } l > k, \tag{5.7}$$

for some $\lambda \in [1/2, \infty)$, *where* C_1 *and* C_2 *are finite positive constants, the sequence*

$$\left\{ \frac{S_n - \beta_n}{\alpha_n} \right\}_{n=1}^{\infty} \tag{5.8}$$

is bounded in probability. Then

$$\sup_n \mathbb{E}\left(\left| \frac{S_n - \beta_n}{\alpha_n} \right|^p \right) < \infty$$

for any $0 < p < 1/\lambda$.

Proof. The proof of this lemma is implicit in the proof of Theorem 6.1 in de Acosta and Giné [1]. ∎

Proof of Theorem 5.1. Introduce $A_n := n^{1/\alpha}\ell(1/n)$ and $B_n := n\int_{\frac{1}{n}}^{1-\frac{1}{n}} Q(u)\,du$ for the normalizing and centering sequences in (5.3) and (5.4), respectively. First we prove that

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \mathbb{P}\left\{ \frac{1}{A_r} \left[\sum_{j=1}^{r} X_j - B_r \right] \le x \right\} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma(x)}{\gamma}\, d\gamma, \tag{5.9}$$

for all $x \in \mathbb{R}$. By the Merge Theorem in (5.3) it suffices to show that

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{G_{\gamma_r}(x)}{r} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma(x)}{\gamma}\, d\gamma, \tag{5.10}$$

for all $x \in \mathbb{R}$. First we infer that (5.10) holds along $\{k_n\}$. Indeed,

$$\begin{aligned}
\frac{1}{\log k_n} \sum_{r=1}^{k_n} \frac{G_{\gamma_r}(x)}{r} &= \frac{1}{\log k_n} \sum_{r=1}^{k_0} \frac{G_{\gamma_r}(x)}{r} + \frac{1}{\log k_n} \sum_{m=1}^{n} \sum_{r=k_{m-1}+1}^{k_m} \frac{G_{\gamma_r}(x)}{r} \\
&= o(1) + \frac{1}{\log k_n} \sum_{m=1}^{n} \sum_{r=k_{m-1}+1}^{k_m} \frac{G_{r/k_m}(x)}{r/k_m} \frac{1}{k_m} \\
&= o(1) + \frac{n \log c}{\log k_n} \frac{1}{\log c} \frac{1}{n} \sum_{m=1}^{n} \sum_{r=k_{m-1}+1}^{k_m} \frac{G_{r/k_m}(x)}{r/k_m} \frac{1}{k_m}
\end{aligned}$$

Here the second equality follows by the fact that $\gamma_r = r/k_m$ for $r \in \{k_{m-1}+1, \ldots, k_m\}$. Introduce

$$\overline{G}_m(s,x) := \begin{cases} \frac{G_{j/k_m}(x)}{j/k_m}, & s \in \left(\frac{j}{k_m}, \frac{j+1}{k_m}\right], \ j = k_{m-1}+1, \ldots, k_m, \\ 0, & \text{elsewhere.} \end{cases}$$

If $\mu \uparrow \lambda \in (0,\infty)$, then, by the right-continuity of $\psi_1^\alpha(\cdot)$ and $\psi_2^\alpha(\cdot)$,

$$V_{0,0}(\psi_1^{\mu,\alpha},\psi_2^{\mu,\alpha},0) \to V_{0,0}(\psi_1^{\lambda,\alpha},\psi_2^{\lambda,\alpha},0),$$

even almost surely. Therefore, $G_\mu(x) \to G_\lambda(x)$ for all $x \in C_{G_\lambda}$. Since $G_\lambda(x)$ is continuous for all $\lambda > 0$ (see Lemma 2.6), this means all $x \in \mathbb{R}$. Hence, for all $x \in \mathbb{R}$, $\overline{G}_n(s,x) \to G_s(x)/s$ for $s \in (1/c, 1]$ and $\overline{G}_n(s,x) \to 0$ for $s \in (0,\infty) \setminus [1/c, 1]$. Thus

$$\sum_{r=k_{m-1}+1}^{k_m} \frac{G_{r/k_m}(x)}{r/k_m}\frac{1}{k_m} = \int_0^\infty \overline{G}_m(s,x)\,ds \to \int_{\frac{1}{c}}^1 \frac{G_\gamma(x)}{\gamma}\,d\gamma,$$

as $m \to \infty$, by Lebesgue's theorem and, since (1.3c) is equivalent to $n\log c/\log k_n \to 1$, all this together yields (5.10) along $\{k_n\}$. But $\log k_{n+1}/\log k_n \to 1$ and the terms $G_{\gamma_n}(x)/\gamma_n$ are non-negative and bounded, thus (5.10) and (5.9) hold along $\{n\} = \mathbb{N}$, as well. It also follows that the limiting distribution function is continuous.

The equivalence of (5.9) and (5.4) will be shown by Theorem 1 of Berkes and Dehling [11]. Therefore, we must check the conditions of that theorem.

For the normalizing constants $A_n = n^{1/\alpha}\ell(1/n)$ in (5.4), by the slow variation of ℓ, there obviously exists a positive constant C for every $\delta \in (0, 1/\alpha)$ such that

$$A_l/A_k \geq C(l/k)^\delta, \qquad l > k,$$

fulfilling the condition (2.2) in [11]. Condition (2.1) of [11] follows by Lemma 5.1, since (5.8) above is obviously implied by Theorem 2.5(i), and (5.7) can be seen by the slow variation of ℓ again. ∎

Lemma 5.2. *Let $X_1, X_2, \ldots$ be independent random variables, $S_n = X_1 + \cdots + X_n$, and let $\{A_n\}$ and $\{B_n\}$ be numerical sequences, $\{A_n\}$ being positive, such that*

$$\frac{1}{\log n}\sum_{r=1}^n \frac{1}{r} I\left\{\frac{S_r - B_r}{A_r} \leq x\right\} \to G(x) \qquad \text{a.s. for all } x \in C_G, \tag{5.11}$$

where G is a distribution function continuous at the origin. Assume that

$$A_l/A_k \geq C_1(l/k)^\beta, \qquad l > k, \tag{5.12}$$

and

$$\mathbb{E}\left(\left|\frac{S_n - B_n}{A_n}\right|^p\right) \leq C_2, \qquad n = 1, 2, \ldots, \tag{5.13}$$

for some positive constants β, p, C_1 and C_2. Then for any positive norming sequence $\{a_n\}$ and any subsequence $\{n_k\} \subset \mathbb{N}$ and (proper or improper) distribution function H the relations

$$\frac{1}{\log n_k}\sum_{r=1}^{n_k} \frac{1}{r} I\left\{\frac{S_r - B_r}{a_r} \leq x\right\} \to H(x) \qquad \text{a.s. for all } x \in C_H, \text{ as } k \to \infty, \tag{5.14}$$

and

$$\frac{1}{\log n_k}\sum_{r=1}^{n_k} \frac{1}{r} \mathbb{P}\left\{\frac{S_r - B_r}{a_r} \leq x\right\} \to H(x) \qquad \text{for all } x \in C_H, \text{ as } k \to \infty, \tag{5.15}$$

are equivalent.

Note before the proof that I can of course be replaced by $\mathbb{P}$ in (5.11) by Theorem 1 of Berkes and Dehling [11] if $\{n_k\} = \mathbb{N}$, where the continuity assumption on G is not needed. The lemma shows that if G is continuous at the origin, then we can replace I by $\mathbb{P}$ even if the norming factor is changed arbitrarily, and this can be done even along subsequences. Note also that the slight continuity condition is satisfied in our applications since the limiting distribution function in (5.4), inheriting this property from the distribution functions $G_\gamma(\cdot)$, $1/c \le \gamma \le 1$, is everywhere continuous.

Proof. First we prove that if $\{a_n\}$ satisfies the additional condition

$$a_n \ge K A_n\,, \quad n = 1, 2, \ldots, \quad \text{for some positive constant } K \tag{5.16}$$

then for any bounded Lipschitz(1) function f on $\mathbb{R}$ we have

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \left[f\left(\frac{S_r - B_r}{a_r}\right) - \mathbb{E}\left(f\left(\frac{S_r - B_r}{a_r}\right)\right)\right] \to 0 \quad \text{a.s. as } n \to \infty. \tag{5.17}$$

Set

$$\xi_r^{(f)} = f(\{S_r - B_r\}/a_r) - \mathbb{E}(f(\{S_r - B_r\}/a_r)).$$

By a standard argument in a.s. central limit theory, (5.17) will follow if we show that

$$\left|\mathbb{E}\left(\xi_k^{(f)} \xi_l^{(f)}\right)\right| \le C \left(\frac{k}{l}\right)^{\rho}, \quad k \le l,$$

for some constants $C > 0$ and $\rho > 0$. Choosing the constant K_1 so that $|f(x)| \le K_1$, $|f(x) - f(y)| \le K_1 |x - y|$ hold for all x, y, we get for any $k \le l$, using (5.12), (5.13) (note that in (5.13) we can assume $p < 1$) and the independence of S_k and $S_l - S_k$,

$$\begin{aligned}
\left|\mathbb{E}\left(\xi_k^{(f)} \xi_l^{(f)}\right)\right| &= \left|\mathrm{Cov}\left(f\left(\frac{S_k - B_k}{a_k}\right), f\left(\frac{S_l - B_l}{a_l}\right)\right)\right| \\
&= \left|\mathrm{Cov}\left(f\left(\frac{S_k - B_k}{a_k}\right), f\left(\frac{S_l - B_l}{a_l}\right) - f\left(\frac{(S_l - S_k) - (B_l - B_k)}{a_l}\right)\right)\right| \\
&\le 4K_1 \,\mathbb{E}\left(K_1 \left|\frac{S_k - B_k}{a_l}\right| \wedge 2K_1\right) \le 8K_1^2 \,\mathbb{E}\left(\left|\frac{S_k - a_k}{K A_l}\right| \wedge 1\right) \\
&\le 8K_1^2 \,\mathbb{E}\left(\left|\frac{S_k - B_k}{K A_l}\right|^{p}\right) \le 8K_1^2 C_2 K^{-p} \left(\frac{A_k}{A_l}\right)^{p} \le C \left(\frac{k}{l}\right)^{\beta p}.
\end{aligned}$$

Dropping now condition (5.16), we first deal with the case $\{n_k\} = \mathbb{N}$; instead of n_k we simply write n. Let $\{a_n\}$ be an arbitrary positive numerical sequence and assume that (5.14) holds. Fix a continuity point $x > 0$ of H (negative x's can be handled similarly). Let $\varepsilon > 0$ and assume that $x \pm \varepsilon$ and $\pm\varepsilon x$ are also continuity points of H (this excludes only countably many ε's). Set

$$a_n^* := \begin{cases} a_n & \text{if} \quad a_n \geq \varepsilon A_n, \\ \varepsilon A_n & \text{if} \quad a_n < \varepsilon A_n, \end{cases}$$

and let $f_{\varepsilon,x}(t)$ denote the bounded Lipschitz(1) function which is 1 for $t \leq x$, 0 for $t \geq x+\varepsilon$ and linear in $[x, x+\varepsilon]$. We consider the six expressions

$$\begin{gathered} \frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}I\left\{\frac{S_r - B_r}{a_r} \leq x\right\}, \quad \frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}I\left\{\frac{S_r - B_r}{a_r^*} \leq x\right\}, \\ \frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}f_{\varepsilon,x}\left(\frac{S_r - B_r}{a_r^*}\right), \quad \frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\mathbb{E}\left(f_{\varepsilon,x}\left(\frac{S_r - B_r}{a_r^*}\right)\right), \\ \frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\mathbb{P}\left\{\frac{S_r - B_r}{a_r^*} \leq x\right\}, \quad \frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\mathbb{P}\left\{\frac{S_r - B_r}{a_r} \leq x\right\}, \end{gathered} \tag{5.18}$$

and estimate the difference between the consecutive ones. Clearly, the difference

$$|I\{(S_r - B_r)/a_r \leq x\} - I\{(S_r - B_r)/a_r^* \leq x\}|$$

equals $I\{a_r x/A_r < (S_r - B_r)/A_r \leq \varepsilon x\} \leq I\{|(S_r - B_r)/A_r| \leq \varepsilon x\}$ for $a_r < \varepsilon A_r$ and thus the difference of the first two expressions in (8) is at most

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}I\left\{\left|\frac{S_r - B_r}{A_r}\right| \leq \varepsilon x\right\}. \tag{5.19}$$

By (5.11), and since $\pm\varepsilon x$ are continuity points of G, the expression in (5.19) converges a.s. to $\psi(\varepsilon) := G(\varepsilon x) - G(-\varepsilon x)$. Since G is continuous at the origin, $\psi(\varepsilon) \to 0$ if $\varepsilon \to 0$. Thus the limsup of the difference of the first two terms in (5.18) is at most $\psi(\varepsilon)$. Replacing I by $\mathbb{P}$, the same estimate applies for the difference of the last two expressions of (5.18). Next we observe that

$$I_{(-\infty,x]} \leq f_{\varepsilon,x} \leq I_{(-\infty,x+\varepsilon]} \tag{5.20}$$

and thus the difference of the second and third expressions in (5.18) is bounded by

$$\left|\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}I\left\{\frac{S_r - B_r}{a_r^*} \leq x\right\} - \frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}I\left\{\frac{S_r - B_r}{a_r^*} \leq x+\varepsilon\right\}\right|. \tag{5.21}$$

Relation (5.14) and the estimate for the difference of the first two expressions in (5.18) imply that for almost every outcome of the sample space and for all sufficiently large n

the first and second terms in (5.18) are within $2\psi(\varepsilon)$ of $H(x)$ and $H(x+\varepsilon)$, respectively, and thus the difference in (5.21) is $\le |H(x+\varepsilon)-H(x)|+4\psi(\varepsilon) =: \psi_1(\varepsilon)$. Since $a_n^* \ge \varepsilon A_n$, the statement proved at the beginning of the proof shows that the difference between the third and fourth term in (5.18) tends to 0 a.s. as $n \to \infty$. Finally, the difference of the fourth and fifth expression in (5.18) can be estimated similarly as the difference of the second and the third, only instead of (5.20) we use the inequality $f_{\varepsilon,x-\varepsilon} \le I_{(-\infty,x]} \le f_{\varepsilon,x}$.

Hence we proved that for sufficiently large n the difference of the first and last expressions in (5.18) is $\le \psi_2(\varepsilon)$ where $\psi_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus (5.14) implies (5.15), at least if $\{n_k\} = \mathbb{N}$. But if $\{n_k\} \ne \mathbb{N}$, then we only have to write n_k in place of n everywhere in (5.18) and apply the same estimates as above, and the implication (5.17)$\Rightarrow$(5.18) follows once again. The converse statement can be proved similarly. ■

Proof of Theorem 5.2. By Lemma 5.2 it suffices to prove that, under the condition of Theorem 5.1, there exists a positive numerical sequence $\{a_n\}$ such that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \mathbb{P}\left\{ \frac{\sum_{j=1}^{r} X_j - r\int_{\frac{1}{r}}^{1-\frac{1}{r}} Q(u)\,du}{a_r} \le x \right\} = J(x), \tag{5.22}$$

for all $x \in C_J$, if and only if $J \in \mathcal{J}^*$.

First assume that (5.22) holds for some $\{a_n\}$ and J. We show that $J \in \mathcal{J}^*$. As in the proof of Theorem 5.1, let $S_n = X_1+\cdots+X_n$, $A_n = n^{1/\alpha}\ell(1/n)$, $B_n = n\int_{\frac{1}{n}}^{1-\frac{1}{n}} Q(u)\,du$ and put $q_n = a_n/A_n$. Let $\{k_n\}$ be a sequence satisfying (1.3c). Then by the Merge Theorem in (5.3),

$$\begin{aligned}
\sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} \mathbb{P}\left\{ \frac{S_r - B_r}{a_r} \le x \right\} &= \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} \mathbb{P}\left\{ \frac{S_r - B_r}{A_r} \le \frac{x a_r}{A_r} \right\} \\
&= \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} G_{\gamma_r}\!\left(\frac{x a_r}{A_r} \right) + o(1) \\
&= \sum_{r=k_{m-1}+1}^{k_m} \frac{G_{r/k_m}(q_r x)}{r} + o(1) \\
&= \int_{k_{m-1}+1}^{k_m} \frac{G_{\lfloor t\rfloor/k_m}(q_{\lfloor t\rfloor} x)}{\lfloor t\rfloor}\,dt + o(1) \\
&= \int_{k_{m-1}+1}^{k_m} \frac{G_{t/k_m}(q_{\lfloor t\rfloor} x)}{t}\,dt + o(1) \\
&= \int_{\frac{1}{c}}^{1} \frac{G_\gamma(q_m(\gamma)x)}{\gamma}\,d\gamma + o(1),
\end{aligned} \tag{5.23}$$

where $q_m(\gamma) = q_{\lfloor k_m\gamma \rfloor}$, with $\lfloor t \rfloor$ standing for the integer part of t, and hence $q_m \in \mathcal{A}$. Now put

$$J_m(x) = \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma(q_m(\gamma)x)}{\gamma}\, d\gamma.$$

Then by (5.22), using also that $\log k_n \sim n \log c$, we have

$$\begin{aligned} J(x) &= \lim_{N\to\infty} \frac{1}{\log k_N} \sum_{m=1}^{N} \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\, \mathbb{P}\left\{ \frac{S_r - B_r}{a_r} \le x \right\} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} J_m(x), \end{aligned}$$

i.e. J is the limit of convex linear combinations of $J_m \in \mathcal{J}$, whence $J \in \mathcal{J}^*$.

We prove the converse statement in several steps.

First assume that $J = J(q\,;x) \in \mathcal{J}$ for some $q \in \mathcal{A}$. Let $\{k_m\}$ satisfy (1.3c). We can find a sequence $\{q_m(\gamma)\}$ of step-functions such that the break-points of $q_m(\gamma)$ are of the form ν/k_m with integer ν, $q_m(\gamma) > 0$ and $q_m(\gamma) \to q(\gamma)$ weakly (pointwise at each continuity point of q). If $(k_{m-1}+1)/k_m < c^{-1}$, then we let $q_m(\gamma) = q_m(1/c)$ for $\gamma \in ((k_{m-1}+1)/k_m, 1/c)$. Put

$$a_r = A_r q_m\!\left(\frac{r}{k_m}\right), \quad k_{m-1}+1 \le r < k_m\,, \quad m = 1, 2, \ldots .$$

Then reading (5.23) backwards, we get

$$\int_{\frac{1}{c}}^{1} \frac{G_\gamma(q_m(\gamma)x)}{\gamma}\, d\gamma = \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\, \mathbb{P}\left\{ \frac{S_r - B_r}{a_r} \le x \right\} + o(1),$$

hence (5.22) holds along the subsequence $\{k_m\}$. This implies, as in the proof of Theorem 5.1, that (5.22) holds along $\{n\} = \mathbb{N}$ as well.

Now let $J(\cdot)$ be of the form

$$J(\cdot) = p_1 J_1(\cdot) + \cdots + p_N J_N(\cdot)$$

with $J_i(\cdot) \in \mathcal{J}$ and let $p_i > 0$, $i = 1, \ldots, N$, be rational numbers such that $\sum_{i=1}^{N} p_i = 1$. By the previous arguments, we can find $a_r^{(i)}$ such that

$$\frac{1}{\log c} \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r}\, \mathbb{P}\left\{ \frac{S_r - B_r}{a_r^{(i)}} \le x \right\} = J_i(x) + o(1), \quad x \in C_{J_i}\,.$$

In particular, suppose that $p_i = \beta_i/\beta$ for some $\beta_i \in \mathbb{N}$, $i = 1, \dots, N$, with $\sum_{i=1}^N \beta_i = \beta$. Define $a_r = a_r^{(i)}$ for all r and m satisfying $k_{m-1}+1 \le r \le k_m$ and $m \equiv j \pmod{\beta}$, where $j \in \mathbb{N}$ is such that $\sum_{l=1}^{i-1} \beta_l \le j \le \sum_{l=1}^{i} \beta_l - 1$, $i = 1, \dots, N$. Then, clearly,

$$\lim_{n\to\infty} \frac{1}{\beta n \log c} \sum_{m=1}^{\beta n} \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} \mathbb{P}\left\{ \frac{S_r - B_r}{a_r} \le x \right\} = p_1 J_1(x) + \cdots + p_N J_N(x) = J(x)$$

for all $x \in C_J$, and so (5.22) holds along the subsequence $\{k_{\beta n}\}$. Thus it also holds along the whole sequence $\{n\} = \mathbb{N}$.

Finally, since the functions $J \in \mathcal{J}_*$ with rational coefficients are weakly dense in $\mathcal{J}^*$, we can find a sequence $\{a_n\}$ such that (5.22) holds. For details on this point, see [7]. This completes the proof of Theorem 5.2. ∎

Proof of Theorem 5.3. The statement concerning the subsequential maximum property of the class $\mathcal{J}^*$ is implicit in the proof of Theorem 5.2: use the literally subsequential original form of Lemma 5.2 and then see (5.23). Now, let $\mathcal{J}_\dagger$ consist of distribution functions of the form

$$J(x) = p_1 J(q_1\,; x) + \cdots + p_N J(q_N\,; x), \quad x \in \mathbb{R},$$

where $N \in \mathbb{N}$, $p_i > 0$, $p_i \in \mathbb{Q}$, $i = 1, \dots, N$, $\sum_{i=1}^N p_i = 1$, and

$$J(q_i\,; x) = \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{G_\gamma(q_i(\gamma)x)}{\gamma}\, d\gamma,$$

where $q_i \in \mathcal{A}$ is a non-negative rational-valued step-function with rational jump-points, $i = 1, \dots, N$. Thus $\mathcal{J}_\dagger$ is a "rational restriction" of $\mathcal{J}_*$. Clearly, $\mathcal{J}_\dagger$ is countable and its weak closure is $\mathcal{J}^*$. Hence it suffices to construct a norming sequence $\{a_n^\star\}$ for which the set of subsequential limits of the averages

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \mathbb{P}\left\{ \frac{\sum_{j=1}^r X_j - r \int_{\frac{1}{r}}^{1-\frac{1}{r}} Q(u)\, du}{a_r^\star} \le x \right\}$$

contains $\mathcal{J}_\dagger$. The construction of such a sequence $\{a_n^\star\}$ can be done by a straightforward modification of the corresponding part in the proof of Theorem 3 in [7]. This completes the proof. ∎

5.3. Almost sure limit theorems for maxima

Retaining the notation from of previous subsection, for $q \in \mathcal{A}$ and any max-semistable distribution function G define

$$K(x) = K(q; x) = \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{1}{\gamma} G^\gamma (q(\gamma)x)\, d\gamma,$$

where the superscript γ in G^γ is simply a power exponent. Define, as above, the class of functions $\mathcal{K} := \{K(q;x) : q \in \mathcal{A}\}$ and let $\mathcal{K}^{(M)}$ be the set of finite convex mixtures of $K \in \mathcal{K}$, i.e. the set of functions of the form

$$p_1 K_1(x) + \cdots + p_N K_N(x), \; p_i \geq 0, \; \sum_{i=1}^{N} p_i = 1, \; K_i \in \mathcal{K}, \; N \geq 1.$$

Finally, let $\mathcal{K}^*$ be the weak closure of $\mathcal{K}^{(M)}$. Whenever speaking of classes $\mathcal{K}$, $\mathcal{K}^{(M)}$ and $\mathcal{K}^*$ it should be always kept in mind that these classes are defined trough a particular max-semistable G and the classes defined trough different G and G' are also different of course.

Theorem 5.4. *Assume that $F \in \mathbb{MD}_{\mathrm{gp}}(G)$ along a subsequence $\{k_n\}$ satisfying (1.3c) with centering and norming constants $\{a_n\}$ and $\{b_n\}$, where G is an arbitrary max-semistable law. Then there exists a numerical sequence $B_n > 0$ such that*

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{M_r - a_r^\circ}{B_r} \leq x \right\} \to K(x), \qquad \text{a.s. for any } x \in C_K, \tag{5.24}$$

if and only if $K \in \mathcal{K}^*$. In particular,

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{M_r - a_r^\circ}{b_r^\circ} \leq x \right\} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{1}{\gamma} G^\gamma(x) \, d\gamma, \tag{5.25}$$

almost surely for all continuity point x of the limiting distribution. Here and in (5.24) the centering and norming sequences $\{a_n^\circ\}$ and $\{b_n^\circ\}$ are as in Theorem 4.4.

There is no point in allowing arbitrary centering sequences in (5.24) since, as shown in Berkes [6], any distribution function can arise this way as an almost sure limiting distribution owing to purely deterministic reasons. Using Corollary 4.1 it can be seen similarly as (5.25) that if $F \in \mathbb{MD}_{\mathrm{gp}}(\Lambda_\nu)$ then

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{M_r - a_r^{(1)}}{b_r^{(1)}} \leq x \right\} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{1}{\gamma} \Lambda_{\gamma\nu}(x + \log\gamma) \, d\gamma, \tag{5.26}$$

a. s. for *all* $x \in \mathbb{R}$, and if $F \in \mathbb{MD}_{\mathrm{gp}}(\Phi_{\alpha,\nu})$ then

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{M_r - a_r^{(2)}}{b_r^{(2)}} \leq x \right\} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{1}{\gamma} \Phi_{\alpha,\gamma\nu}\left(\gamma^{1/\alpha} x\right) d\gamma, \tag{5.27}$$

a. s. for *all* $x \in \mathbb{R}$, and, finally, if $F \in \mathbb{MD}_{\mathrm{gp}}(\Psi_{\alpha,\nu})$ then

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{ \frac{M_r - a_r^{(3)}}{b_r^{(3)}} \leq x \right\} \to \frac{1}{\log c} \int_{\frac{1}{c}}^{1} \frac{1}{\gamma} \Psi_{\alpha,\gamma\nu}\left(\gamma^{-1/\alpha} x\right) d\gamma, \tag{5.28}$$

a. s. for *all* $x \in \mathbb{R}$. Observe that all the distribution functions figuring on the right-hand sides of (5.26)–(5.28) are always continuous, unlike the one in (5.25). We note that Theorem 5.4 is essentially different from and not implied by the results in Chuprunov [18].

It was shown in Section 4.3 that all (non-degenerate) subsequential weak limits of maxima from the max-domain of geometric partial attraction of a max-semistable law G are within the family G. Our last theorem, an analogue of Theorem 5.3 and of Theorem 3 in Berkes and Csáki [7], states that this subsequential closedness is preserved for almost sure limits as well, perhaps in an even stronger sense.

Theorem 5.5. *Suppose the conditions of Theorem 5.4. For any subsequence* $\{n_k\}_{k=1}^{\infty}$

$$\lim_{k\to\infty} \frac{1}{\log n_k} \sum_{r=1}^{n_k} \frac{1}{r} I\left\{\frac{M_r - a_r^\circ}{B_r} \le x\right\} = K(x), \qquad \text{a.s. for any } x \in C_K,$$

implies $K \in \mathcal{K}^*$. *Furthermore, there exists a 'universal' norming sequence* $B_n^\star$ *such that the set of almost sure subsequential limits of the sequence*

$$\frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} I\left\{\frac{M_r - a_r^\circ}{B_r^\star} \le x\right\}$$

coincides with the entire $\mathcal{K}^*$.

The questions described in the previous section concerning the class $\mathcal{J}^*$ arise in connection with the class $\mathcal{K}^*$, as well, and they are similarly open.

Proofs

The proofs of the theorems above require two lemmas.

Lemma 5.3. *Let* $\{X_n\}$ *be a sequence of independent identically distributed random variables. Then for any subsequence* $\{n_k\} \subset \mathbb{N}$ *and (proper or improper) distribution function* K *and any real sequences* $\{a_n\}$ *and* $\{b_n\}$ *the relations*

$$\lim_{k\to\infty} \frac{1}{\log n_k} \sum_{r=1}^{n_k} \frac{1}{r} \mathbb{P}\left\{\frac{M_r - a_r}{b_r} \le x\right\} = K(x), \qquad \text{for any } x \in C_K, \tag{5.29}$$

and

$$\lim_{k\to\infty} \frac{1}{\log n_k} \sum_{r=1}^{n_k} \frac{1}{r} I\left\{\frac{M_r - a_r}{b_r} \le x\right\} = K(x), \qquad \text{a.s. for any } x \in C_K, \tag{5.30}$$

are equivalent.

Proof. Introduce $M_r^* := M_r - a_r)/b_r$. It follows exactly the same way as in the proof of Theorem B in [8] (cf. also Lemma 2 in [9]) that for any bounded Lipschitz(1) function f on $\mathbb{R}$ we have

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\left[f(M_r^*) - \mathbb{E}(f(M_r^*))\right] \to 0 \quad \text{a.s.} \tag{5.31}$$

as $n \to \infty$.

We assume first (5.29) and show (5.30). Fix some $x \in C_\Lambda$ and choose a sequence $\varepsilon_m \to 0$, $\varepsilon_m > 0$, such that $x \pm \varepsilon_m \in C_\Lambda$ for each $m \in \mathbb{N}$. Let the bounded Lipschitz(1) functions $f_m^x(t)$ and $g_m^x(t)$, $t \in \mathbb{R}$, be defined by

$$f_m^x(t) := \begin{cases} 1, & \text{if } t \le x - \varepsilon_m, \\ 0, & \text{if } t \ge x, \\ \text{linear in between,} & \end{cases}$$

and

$$g_m^x(t) := \begin{cases} 1, & \text{if } t \le x, \\ 0, & \text{if } t \ge x + \varepsilon_m, \\ \text{linear in between.} & \end{cases}$$

Clearly, $f_m^x(t) \le I(t \le x) \le g_m^x(t)$ for all $t \in \mathbb{R}$, and hence

$$\frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{f_m^x(M_r^*) - \mathbb{E}(g_m^x(M_r^*))}{r} \le \theta_k(x) \le \frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{g_m^x(M_r^*) - \mathbb{E}(f_m^x(M_r^*))}{r} \tag{5.32}$$

for each $k, m \in \mathbb{N}$, where

$$\theta_k(x) = \frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{I(M_r^* \le x) - \mathbb{P}\{M_r^* \le x\}}{r}.$$

Denoting by $\xi_k^m(x)$ the lower bound in (5.32), we have

$$\begin{aligned}
\limsup_{k\to\infty}\xi_k^m(x) &= \limsup_{k\to\infty}\frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{f_m^x(M_r^*) - \mathbb{E}(f_m^x(M_r^*))}{r} \\
&\quad + \limsup_{k\to\infty}\frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{\mathbb{E}(f_m^x(M_r^*)) - \mathbb{E}(g_m^x(M_r^*))}{r} \\
&\ge 0 + \limsup_{k\to\infty}\frac{1}{\log n_k}\sum_{r=1}^{n_k}\frac{\mathbb{P}\{M_r^* \le x - \varepsilon_m\} - \mathbb{P}\{M_r^* \le x + \varepsilon_m\}}{r} \\
&= \Lambda(x - \varepsilon_m) - \Lambda(x + \varepsilon_m),
\end{aligned}$$

which goes to 0 as $m \to \infty$. Here the inequality holds almost surely for each $m \in \mathbb{N}$ by virtue of (5.31), and hence it holds on a set of probability 1 for all $m \in \mathbb{N}$. A similar estimate for the right-hand side of (5.32) shows that $\theta_k(x) \to 0$ a.s. as $k \to 0$, that is, (5.30) follows from (5.29) as claimed. Assuming (5.30), the converse implication follows by taking expectation. ■

The second lemma can be proved in an entirely similar way as the corresponding result with sums in place of maxima in the proof of Theorem 2 in Berkes and Csáki [7].

Lemma 5.4. *Using the notation of Lemma 5.3, suppose that for a distribution function K and a sequence of distribution functions K_m*

$$\mathcal{L}(K, K_m) \to 0, \qquad \text{as } m \to \infty,$$

and for some centering and norming sequences $\{a_{m,n}\}_{n=1}^{\infty}$ and $\{b_{m,n}\}_{n=1}^{\infty}$

$$K_m(x) = \lim_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \mathbb{P}\left\{ \frac{M_r - a_{m,r}}{b_{m,r}} \le x \right\},$$

for all $x \in C_{K_m}$: both K and K_m may be degenerate. Then there exist centering and norming sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that

$$K(x) = \lim_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \mathbb{P}\left\{ \frac{M_r - a_r}{b_r} \le x \right\},$$

for all $x \in C_K$. ∎

Proof of Theorem 5.4. By Lemma 5.3 it is sufficient to prove that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{r=1}^{n} \frac{1}{r} \mathbb{P}\left\{ \frac{M_r - a_r^\circ}{B_r} \le x \right\} = K(x), \qquad \text{for any } x \in C_K, \tag{5.33}$$

if and only if $K \in \mathcal{K}^*$. First we deal with necessity. Recall the notation of Theorem 4.4 and put $q_n := B_n / b_n^\circ$. Introduce

$$N_j := \left\{ n \in \mathbb{N} : q_n < j^{-1} \right\}, \qquad j \in \mathbb{N},$$

and

$$L_j := \limsup_{n\to\infty} \frac{1}{\log k_n} \sum_{r=1}^{k_n} \frac{1}{r} I(r \in N_j), \qquad j \in \mathbb{N}.$$

Obviously $\{N_j\}$ is a decreasing sequence of sets, hence $L_1 \ge L_2 \ge \cdots$. Let $L := \lim_{j\to\infty} L_j$. We construct a subsequence $n(l)$ by the following rule:

$$n(0) := 1, \qquad n(l) := \min\left\{ n > n(l-1) : \frac{1}{k_n} \sum_{r=1}^{k_n} \frac{1}{r} I(r \in N_j) \ge L - \frac{1}{l} \right\}, \quad l \in \mathbb{N}.$$

This sequence satisfies for any $j \in \mathbb{N}$

$$L_j \ge \limsup_{l\to\infty} \frac{1}{\log k_{n(l)}} \sum_{r=1}^{k_{n(l)}} \frac{1}{r} I(r \in N_j) \ge \liminf_{l\to\infty} \frac{1}{\log k_{n(l)}} \sum_{r=1}^{k_{n(l)}} \frac{1}{r} I(r \in N_j) \ge L.$$

The first inequality is true for *any* subsequence and the last one holds since for $l \geq j$

$$\frac{1}{\log k_{n(l)}} \sum_{r=1}^{k_{n(l)}} \frac{1}{r} I(r \in N_j) \geq \frac{1}{\log k_{n(l)}} \sum_{r=1}^{k_{n(l)}} \frac{1}{r} I(r \in N_l) \geq L - \frac{1}{l}.$$

We have to prove that for any given $\varepsilon^* > 0$ there exists a $K' \in \mathcal{K}^{(M)}$ such that $\mathcal{L}(K', K) \leq \varepsilon^*$. To show the existence of such a K' we will need the quantities $V, W \in \mathbb{N}$, $W > V$ and $\varepsilon, \delta > 0$. These quantities all depend on ε^* and their choice will be made later. (At this point, one should only think of V and W as large integers and of ε and δ as small positive numbers.) Introduce also

$$q_r^V = \begin{cases} q_r, & q_r \geq V^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

There exists a threshold number $N_{\varepsilon,\delta}$ depending on δ and ε such that for $m \geq N_{\varepsilon,\delta}$ and for all $x > -\varepsilon W$,

$$\begin{aligned}
&\sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} \mathbb{P}\left\{ \frac{M_r - a_r^\circ}{B_r} \leq x \right\} = \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} \mathbb{P}\left\{ \frac{M_r - a_r^\circ}{b_r^\circ} \leq \frac{xB_r}{b_r^\circ} \right\} \\
&\geq \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} G^{\gamma_r}\left(\frac{xB_r}{b_r^\circ} - \varepsilon \right) - \varepsilon \\
&\geq \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} G^{\gamma_r}\left(q_r^V(x - \varepsilon V)\right) \\
&\qquad + \sum_{r=k_{m-1}+1}^{k_m} \frac{I(r \in N_V)}{r} \left[G^{\gamma_r}\left(\frac{xB_r}{b_r^\circ} - \varepsilon \right) - G^{\gamma_r}(0) \right] - \varepsilon \\
&\geq \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} G^{\gamma_r}\left(q_r^V(x - \varepsilon V)\right) - \sum_{r=k_{m-1}+1}^{k_m} \frac{I(r \in N_V \setminus N_W)}{r} \\
&\qquad - \sum_{r=k_{m-1}+1}^{k_m} \frac{I(r \in N_W)}{r} \left[G^{\gamma_r}(0) - G^{\gamma_r}(-2\varepsilon) \right] - \varepsilon \\
&=: \int_{k_{m-1}+1}^{k_m} \frac{1}{\lfloor t \rfloor} G^{\lfloor t \rfloor / k_m}\left(q_{\lfloor t \rfloor}^V(x - \varepsilon V)\right) dt - \Delta_1(m, V, W) - \Delta_2(m, W, \varepsilon) - \varepsilon \\
&\geq \int_{k_{m-1}+1}^{k_m} \frac{1}{t} G^{t/k_m}\left(q_{\lfloor t \rfloor}^V(x - \varepsilon V)\right) dt - \Delta_1(m, V, W) - \Delta_2(m, W, \varepsilon) - \varepsilon - \frac{\delta}{2} \\
&\geq \int_{\frac{1}{c}}^{1} \frac{1}{\gamma} G^{\gamma}\left(q_m^V(\gamma)(x - V\varepsilon)\right) d\gamma - \Delta_1(m, V, W) - \Delta_2(m, W, \varepsilon) - \varepsilon - \delta,
\end{aligned}$$

where $q_m^V(\gamma) = q^V_{\lfloor k_m \gamma \rfloor}$, and hence $q_m^V \in \mathcal{A}$. The first inequality in the second row holds by Theorem 4.4 and condition $x > -\varepsilon W$ is only used when obtaining the third one.

Define

$$K_m^V(x) = K(q_m^V, x) = \frac{1}{\log c} \int_{\frac{1}{c}}^1 \frac{1}{\gamma} G^\gamma\big(q_m^V(\gamma)x\big)\, d\gamma \in \mathcal{K}.$$

Relation (5.33) applied along $\{k_{n(l)}\}$ and $\log k_n \sim n \log c$ entails

$$K(x) = \lim_{l\to\infty} \frac{1}{n(l)} \sum_{m=1}^{n(l)} \frac{1}{\log c} \sum_{r=k_{m-1}}^{k_m} \frac{1}{r} \mathbb{P}\left\{ \frac{M_r - a_r^\circ}{B_r} \leq x \right\}, \qquad \text{for any } x \in C_K.$$

Hence by the above calculation for any $x \in C_K$, $x > -\varepsilon W$ we have

$$\begin{aligned} K(x) \geq{} & \limsup_{l\to\infty} \frac{1}{n(l)} \sum_{m=1}^{n(l)} K_m^V(x - \varepsilon V) - \limsup_{l\to\infty} \frac{1}{n(l)\log c} \sum_{m=1}^{n(l)} \Delta_1(m, V, W) \\ & - \limsup_{l\to\infty} \frac{1}{n(l)\log c} \sum_{m=1}^{n(l)} \Delta_2(m, W, \varepsilon) - \frac{(\varepsilon + \delta)}{\log c}. \end{aligned}$$

Here

$$\begin{aligned} & \limsup_{l\to\infty} \frac{1}{n(l)\log c} \sum_{m=1}^{n(l)} \Delta_1(m, V, W) \\ & \qquad = \limsup_{l\to\infty} \frac{1}{\log k_{n(l)}} \sum_{r=1}^{k_{n(l)}} \frac{I(r \in N_V \setminus N_W)}{r} \leq L_V - L, \end{aligned}$$

where the upper bound is independent of W, and

$$\begin{aligned} & \limsup_{l\to\infty} \frac{1}{n(l)\log c} \sum_{m=1}^{n(l)} \Delta_2(m, W, \varepsilon) \\ & = \limsup_{l\to\infty} \frac{1}{\log k_{n(l)}} \sum_{r=1}^{k_{n(l)}} \frac{I(r \in N_W)}{r} \left[G^{\gamma_r}(0) - G^{\gamma_r}(-2\varepsilon) \right] \leq G^{\frac{1}{c}}(2\varepsilon) - G^{\frac{1}{c}}(-2\varepsilon). \end{aligned}$$

Let now $V \in \mathbb{N}$, $\varepsilon, \delta > 0$ be chosen so that

$$L_V - L < \frac{\varepsilon^*}{8}, \qquad G^{\frac{1}{c}}(2\varepsilon) - G^{\frac{1}{c}}(-2\varepsilon) < \frac{\varepsilon^*}{8}, \qquad \frac{\varepsilon + \delta}{\log c} < \frac{\varepsilon^*}{4}, \quad \text{and} \quad \varepsilon V < \frac{\varepsilon^*}{2}$$

jointly hold. Then for every $W > V$ and $x > -\varepsilon W$ by the calculations above we have

$$\limsup_{l\to\infty} \frac{1}{n(l)} \sum_{m=1}^{n(l)} K_m^V\left(x - \frac{\varepsilon^*}{2}\right) - \frac{\varepsilon^*}{2} < K(x).$$

This inequality is independent of W, hence it is valid for all $x \in \mathbb{R}$. This yields

$$\begin{aligned}\limsup_{l\to\infty} \frac{1}{n(l)} \sum_{m=1}^{n(l)} K_m^V\Big(x - \frac{\varepsilon^*}{2}\Big) - \frac{\varepsilon^*}{2} < K(x) \\ < \liminf_{l\to\infty} \frac{1}{n(l)} \sum_{m=1}^{n(l)} K_m^V\Big(x + \frac{\varepsilon^*}{2}\Big) + \frac{\varepsilon^*}{2},\end{aligned} \tag{5.34}$$

where the upper bound follows similarly as the lower one. Choose now a finite sequence $\{x_j\}_{j=0}^M$ of real numbers with the following properties:

$$K(x_0) \leq \lim_{x\to-\infty} K(x) + \frac{\varepsilon^*}{2}, \quad K(x_M) \geq \lim_{x\to\infty} K(x) - \frac{\varepsilon^*}{2}, \quad x_j = x_{j-1} + \frac{\varepsilon^*}{2},\ j = 1, \dots M.$$

By (5.34) there exists $L^* \in \mathbb{N}$ such that the inequalities

$$\frac{1}{n(L^*)} \sum_{m=1}^{n(L^*)} K_m^V\Big(x_j - \frac{\varepsilon^*}{2}\Big) - \frac{\varepsilon^*}{2} \leq K(x_j) \leq \frac{1}{n(L^*)} \sum_{m=1}^{n(L^*)} K_m^V\Big(x_j + \frac{\varepsilon^*}{2}\Big) + \frac{\varepsilon^*}{2} \tag{5.35}$$

hold true for all $j \in \{0, \dots, M\}$. The choice of $\{x_j\}$ ensures that (5.35) remains valid for all $x \in \mathbb{R}$ if $\varepsilon^*/2$ is replaced by ε^*. However, this is nothing but

$$\mathcal{L}\left(\frac{1}{n(L^*)} \sum_{m=1}^{n(L^*)} K_m^V(\cdot), K(\cdot)\right) \leq \varepsilon^*,$$

where $\frac{1}{n(L^*)} \sum_{m=1}^{n(L^*)} K_m^V(\cdot)$ is a convex linear combination of functions from $\mathcal{K}$, hence it is in $\mathcal{K}^{(M)}$. Necessity is proved.

Turning to sufficiency, define the set $\mathcal{K}_+ \subset \mathcal{K}$ as the set of functions of the form $K = K(q; x)$ with $q \in \mathcal{A}$ and $q(\gamma) > 0$, $\gamma \in [\frac{1}{c}, 1]$. Obviously, $\mathcal{K}_+$ is dense in $\mathcal{K}$.

Suppose first that $K = K(q; x) \in \mathcal{K}_+$. We can find a sequence $\{q_m(\gamma)\}$ of step-functions such that the breakpoints of $q_m(\gamma)$ are of the form μ/k_m with integer μ, $q_m(\gamma) \to q(\gamma)$ weakly (pointwise at each continuity point of q), and there exists $V_* \in \mathbb{N}$ such that $q_m(\gamma) > V_*^{-1}$, $\gamma \in [\frac{1}{c}, 1]$. (We may suppose $V_* \geq 3$.) If $(k_{m-1}+1)/k_m < c^{-1}$, then we let $q_m(\gamma) := q_m(1/c)$ for $\gamma \in ((k_{m-1}+1)/k_m, 1/c)$. Put

$$B_r = q_m\left(\frac{r}{k_m}\right) b_r^\circ, \qquad k_{m-1} + 1 \leq r < k_m,\ m = 1, 2, \dots .$$

Observe that the error terms Δ_1 and Δ_2 in the long calculation in the proof of the necessity part vanish if $V^{-1} < B_r/b_r^\circ$, $r = k_{m-1}+1, \dots, k_m$. Hence reading this calculation backwards for arbitrary $\varepsilon > 0$ with $V = V_*$ and $\delta = \varepsilon$ yields

$$\begin{aligned}\int_{\frac{1}{c}}^{1} \frac{1}{\gamma} G^{\gamma}\big(q_m(\gamma)(x - \varepsilon V_*)\big)\, d\gamma - \varepsilon V_* &< \sum_{r=k_{m-1}+1}^{k_m} \frac{1}{r} \mathbb{P}\Big\{\frac{M_r - a_r^\circ}{B_r} \le x\Big\} \\ &< \int_{\frac{1}{c}}^{1} \frac{1}{\gamma} G^{\gamma}\big(q_m(\gamma)(x + \varepsilon V_*)\big)\, d\gamma + \varepsilon V_*,\end{aligned}$$

for all $x \in \mathbb{R}$ provided $m \ge N_\varepsilon$, where $N_\varepsilon \in \mathbb{N}$ is a threshold number depending only on ε. This in turn implies that there exists $N_\varepsilon^* \in \mathbb{N}$ such that for $n > N_\varepsilon^*$,

$$\mathcal{L}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\log c}\int_{\frac{1}{c}}^{1}\frac{1}{\gamma}G^{\gamma}(q_m(\gamma)x)\,d\gamma,\ \frac{1}{\log k_n}\sum_{r=1}^{k_n}\frac{1}{r}\mathbb{P}\Big\{\frac{M_r - a_r^\circ}{B_r} \le x\Big\}\right\} \le \varepsilon V_*.$$

However, since ε was arbitrary and V_* is fixed, this means that the Lévy-distance on the left-hand side converges to 0 as $n \to \infty$. Since

$$\mathcal{L}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\log c}\int_{\frac{1}{c}}^{1}\frac{1}{\gamma}G^{\gamma}(q_m(\gamma)x)\,d\gamma,\ \frac{1}{\log c}\int_{\frac{1}{c}}^{1}\frac{1}{\gamma}G^{\gamma}(q(\gamma)x)\,d\gamma\right\} \to 0, \qquad \text{as } n \to \infty,$$

it follows via the triangle inequality and (5.33) that (5.24) holds along $\{k_n\}$. Taking into account that $\frac{\log k_{n+1}}{\log k_n} \to 1$, it also holds along the entire $\{n\}$. This consideration proves (5.25) as well, since in this case $q(\gamma) \equiv 1$. Define now the set $\mathcal{K}_{\mathbb{Q},+}^{(M)}$ as the set of distribution functions of the form

$$K = p_1 K_1 + \ldots + p_N K_N$$

with $K_i \in \mathcal{K}_+$ where p_i, $i = 1, \ldots, N$, are rational numbers, $\sum_{i=1}^{N} p_i = 1$. By the previous arguments we can find a norming sequence $B_{i,r}$ such that for any fixed $\varepsilon > 0$

$$\mathcal{L}\left\{\frac{1}{\log c}\sum_{r=k_{m-1}+1}^{k_m}\frac{1}{r}\mathbb{P}\Big(\frac{M_r - a_r^\circ}{B_{i,r}} \le x\Big),\ K_i(x)\right\} \le \varepsilon, \qquad i = 1, \ldots, N,$$

provided $m \ge N_\varepsilon$, where N_ε is a threshold number depending on ε.

Let $p_i := \beta_i/\beta$, $\beta_i \in \mathbb{N}$, with $\sum_i \beta_i = \beta$. For $m \equiv j \pmod{\beta}$, $\sum_{l=1}^{i-1}\beta_l \le j \le \sum_{l=1}^{i}\beta_l - 1$ define $B_r = B_{i,r}$, $k_{m-1} + 1 \le r \le k_m$, $i = 1, \ldots, N$. Then clearly

$$\mathcal{L}\left\{\frac{1}{\beta n \log c}\sum_{m=1}^{\beta n}\sum_{r=k_{m-1}+1}^{k_m}\frac{1}{r}\mathbb{P}\Big(\frac{M_r - a_r^\circ}{B_r} \le x\Big), K(x)\right\} \to 0, \qquad \text{as } n \to \infty,$$

and thus (5.24) holds along $\{k_{\beta n}\}$, hence it holds along $\{n\} = \mathbb{N}$, too. Since $\mathcal{K}_{\mathbb{Q},+}^{(M)}$ is weakly dense in $\mathcal{K}^{(M)}$, it is weakly dense in $\mathcal{K}^*$ just as well. The sufficiency part of the theorem now follows from Lemma 5.4. ∎

Proof of Theorem 5.5. The proof of the first statement of the theorem is entirely similar to that of the necessity part of Theorem 5.4. As to the second, introduce $\mathcal{A}_{\mathbb{Q}} \subset \mathcal{A}$, the set of rational-valued step functions with rational jump points and let $\mathcal{K}_{\mathbb{Q}}$ consist of distribution functions of the form $K(q,x)$ with $q \in \mathcal{A}_{\mathbb{Q}}$. Finally, denote by $\mathcal{K}_{\mathbb{Q},\mathbb{Q}}^{(M)}$ the set of finite convex linear combinations with rational coefficients of $K \in \mathcal{K}_{\mathbb{Q}}$. Clearly, the weak closure of $\mathcal{K}_{\mathbb{Q}}$ and of $\mathcal{K}_{\mathbb{Q},\mathbb{Q}}^{(M)}$ is $\mathcal{K}$ and $\mathcal{K}^*$, respectively, and, most importantly, $\mathcal{K}_{\mathbb{Q},\mathbb{Q}}^{(M)}$ is countable. Hence, using Lemma 5.3 and Lemma 5.4, it suffices to find a norming sequence $\{B_n^\star\}$ such that the weak limits of

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\,\mathbb{P}\left\{\frac{M_r-a_r^\circ}{B_r^\star}\le x\right\}$$

coincide with the entire $\mathcal{K}_{\mathbb{Q},\mathbb{Q}}^{(M)}$. By the sufficiency part of the previous theorem for each $K_i \in \mathcal{K}_{\mathbb{Q},\mathbb{Q}}^{(M)}$ there exists a norming sequence $\{B_{i,r}\}$ such that

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\,\mathbb{P}\left\{\frac{M_r-a_r^\circ}{B_{i,r}}\le x\right\}\to K_i,$$

for each $x \in C_{K_i}$. Choose now a sequence of distribution functions $\{K_i\}_{i=1}^{\infty} \subset \mathcal{K}_{\mathbb{Q},\mathbb{Q}}^{(M)}$ such that each element of $\mathcal{K}_{\mathbb{Q},\mathbb{Q}}^{(M)}$ is contained in $\{K_i\}$ infinitely many times. Denote by $\{B_{i,n}\}$ the norming sequence pertaining to K_i. If $1 = N_1 < N_2 < \cdots$ is a sequence of positive integers satisfying $\log N_{k+1}/\log N_k \to \infty$, as $k \to \infty$, then

$$B_n^\star := B_{i,n}, \qquad \text{for } n \in \{N_i,\ldots,N_{i+1}-1\},$$

is an appropriate choice. ∎

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Contents

Summary

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with a common distribution function $F(x) := \mathbb{P}\{X_1 \leq x\}$, $x \in \mathbb{R}$, and for each positive integer $n \in \mathbb{N}$ introduce $X_{1,n} \leq \ldots \leq X_{n,n}$, the order statistics based on $X_1, \ldots, X_n$. We say that the distribution function G is semistable and F is in the domain of geometric partial attraction of G, which we denote by $F \in \mathbb{D}_{\mathrm{gp}}(G)$, if there exists a subsequence $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ satisfying the growth condition that

$$\lim_{n\to\infty} k_{n+1}/k_n = c, \qquad 1 \leq c < \infty,$$

furthermore, centring and norming sequences $B_{k_n} \in \mathbb{R}$ and $A_{k_n} > 0$ such that

$$\frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{k_n} X_j - B_{k_n} \right\} \xrightarrow{\mathcal{D}} V,$$

with $\mathbb{P}(V \leq x) = G(x)$. Here and in the sequel every convergence statement is meant as $n \to \infty$ and the sign $\xrightarrow{\mathcal{D}}$ is used to denote convergence in distribution. Semistable laws form a subclass of infinitely divisible distributions, and this subclass is wider than that of stable laws. (Stable laws arise if $k_n = n$.)

In the dissertation we use the "probabilistic approach" of Csörgő, Haeusler, and Mason. In Theorem 2.1 we give a "probabilistic" representation of semistable laws, and in Theorem 2.3 we characterize the quantile functions of the attracted distributions, the corresponding characterization of the distribution functions being given in Corollary 2.3. The conditions are necessary and sufficient. In Theorem 2.4 we show by an example that the domain of geometric partial attraction of a semistable law is a proper subset of the ordinary domain of partial attraction, which arises if the subsequence k_n is allowed to be arbitrary.

In Theorem 2.5 we determine the subsequences along which sums of independent, identically distributed random variables with a common distribution function F converge if G is a semistable, but not stable law and $F \in \mathbb{D}_{\mathrm{gp}}(G)$. The result implies the stochastic compactness of an $F \in \mathbb{D}_{\mathrm{gp}}(G)$. Theorem 2.6 is the Merge Theorem. Merge is a central notion connected to semistability: if $F \in \mathbb{D}_{\mathrm{gp}}(G)$ for a semistable G and F_n denotes the distribution function of the suitably centred and normed sum $S_n(0,0) := \sum_{j=1}^{n} X_j$, then typically the sequence F_n does not converge, only F_{k_n} does. Yet the members of F_n merge together with semistable distribution functions around G. This, and many of the results above, holds in the case of the *lightly trimmed* sums $S_{k_n}(l, m) := \sum_{j=l+1}^{k_n - m} X_{j,k_n}$, as well, where l and m are arbitrary two fixed integers. An advantage of the "probabilistic approach" taken is that it is capable to deal with both full and lightly trimmed sums in a unified frame.

In Section 3 we deal with the convergence of the *moderately trimmed sums* $S_n(l_n, m_n) = \sum_{j=l_n+1}^{n-m_n} X_{j,n}$, where l_n and m_n are two sequences of integers such that $l_n \to \infty$, $\frac{l_n}{n} \to 0$, and $m_n \to \infty$, $\frac{m_n}{n} \to 0$. Our main result in Theorem 3.1 is that if G is a not completely asymmetric semistable law satisfying a mild continuity condition, $F \in \mathbb{D}_{\mathrm{gp}}(G)$, then $S_n(l_n, m_n)$ is asymptotically normal for any choice of l_n and m_n. If the continuity condition is violated, we show in Theorem 3.2 that asymptotic normality can still be achieved with a suitable choice of l_n and m_n. In Theorem 3.3 we show that asymptotic normality of the full sum is due to an arbitrarily small portion of upper and lower order statistics, since the middle of the sample vanishes by what is needed to normalize the full sum.

In Section 4 we deal with max-semistable laws and their max-domains of geometric partial attraction. Introducing $M_n := X_{n,n} = \max\{X_1, \dots, X_n\}$, we say that G is max-semistable and that F belongs to the max-domain of geometric partial attraction of G, written $F \in \mathbb{MD}_{\mathrm{gp}}(G)$, if for suitable sequences of centring and norming constants a_n and $b_n > 0$

$$\mathbb{P}(b_n^{-1}(M_{k_n} - a_n) \le x) = \Big(F\big(b_n x + a_n\big)\Big)^{k_n} \to G(x), \qquad x \in C_G,$$

for a subsequence $\{k_n\}$ that satisfies the growth condition above. Here C_G denotes the set of continuity points of G. We know that modulo a scale and location transformation max-semistable laws belong to one of the following classes:

$$\begin{aligned}
\Lambda_\nu(x) &= \exp\big(-\exp(-x)\nu(x)\big), \quad && x \in \mathbb{R}, \\
\Phi_{\alpha,\nu}(x) &= \exp\big(-x^{-\alpha}\nu(\log x)\big), && x \in (0,\infty), \\
\Psi_{\alpha,\nu}(x) &= \exp\Big(-|x|^{\alpha}\nu\big(\log|x|\big)\Big), && x \in (-\infty, 0),
\end{aligned}$$

where $\nu(\cdot)$ is a positive periodic function. In Theorems 4.1–4.3 we give a mathematically complete characterization of max-domains of geometric partial attraction and we give a counterexample to point out the fault of the previous characterization that can be found in the literature.

Theorem 4.4 is the Merge Theorem for Maxima. We have seen that merging together of sums holds in the uniform metric, however, for maxima it holds only in the Lévy distance: but this suits most purposes just as well. In Theorem 4.6 we determine the subsequences along which convergence takes place: the conditions are necessary and sufficient again.

In Section 5, which deals with almost sure limit theorems, we give an application of the merge theorems. Let $A_n > 0$ and B_n be two sequences of numbers, $S_n = X_1 + \cdots + X_n$, and let the indicator function be denoted by $I(\cdot)$. Many authors investigated the connection between the weak limit theorem

$$\mathbb{P}\left\{\frac{S_n - B_n}{A_n} \le x\right\} \to G(x), \qquad \text{for any } x \in C_G,$$

and the corresponding "strong", almost sure result

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}I\left\{\frac{S_r-B_r}{A_r}\le x\right\}\to G(x),\qquad \text{a.s. for any } x\in C_G.$$

It turns out that under mild conditions the "weak" statement actually implies the strong one. In Theorem 5.1 of the dissertation we show that for a wide class of distributions, namely, for those in the geometric partial attraction of semistable and max-semistable laws, an almost sure, strong limit theorem holds *along the entire* $\mathbb{N}$ despite the fact that the corresponding weak theorem does not. In fact, it may not: it belongs to the essence of semistability that we have convergence only along $\{k_n\}$, and not along $\{n\}$. In Theorem 5.2 we determine the set $\mathcal{J}^*$ of possible almost sure limits if the norming constants that are given by the theory developed so far are changed to an arbitrary numerical sequence. In Theorem 5.3 we show that the class obtained in this way is closed under taking subsequential limits, furthermore, there exists a "universal" norming sequence such that the set of the arising partial limits is the whole $\mathcal{J}^*$.

In Theorems 5.4 and 5.5 we prove analogous statements for maxima in place of sums. It should be emphasised that although the results for semistable and max-semistable laws show up a remarkable mathematical symmetry, the tools applied are entirely different. In Theorem 5.4 we prove the almost sure statement for general norming sequences and in Theorem 5.5 we show that the arising class $\mathcal{K}^*$ of possible almost sure limits is closed again under taking subsequential limits. We prove that also in this case there exists a "universal" norming sequence such that the set of the arising partial limits is the whole $\mathcal{K}^*$.

Összefoglaló

Határeloszlástételek a szemistabilis és max-szemistabilis eloszlások geometriai parciális vonzástartományához tartozó változók összegeire és maximumaira

Legyenek $X_1, X_2, \ldots$ független azonos eloszlású véletlen változók egy $(\Omega, \mathcal{A}, \mathbb{P})$ valószínűségi mezőn, közös $F(x) := \mathbb{P}\{X_1 \leq x\}$ (egydimenziós) eloszlásfüggvénnyel, és jelölje $X_{1,n} \leq \ldots \leq X_{n,n}$ az $X_1, \ldots, X_n$-ből képzett rendstatisztikákat. Akkor mondjuk, hogy a G eloszlásfüggvény *szemistabilis*, az F pedig a G geometriai parciális vonzástartományához tartozik, jelben $F \in \mathbb{D}_{\mathrm{gp}}(G)$, ha létezik olyan $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ részsorozat, amely kielégíti a

$$\lim_{n \to \infty} k_{n+1}/k_n = c, \qquad 1 \leq c < \infty,$$

növekedési feltételt, valamint $B_{k_n} \in \mathbb{R}$ és $A_{k_n} > 0$ centralizáló és normalizáló konstansok úgy, hogy

$$\frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{k_n} X_j - B_{k_n} \right\} \xrightarrow{\mathcal{D}} V,$$

ahol $\mathbb{P}(V \leq x) = G(x)$. Itt és a továbbiak során minden konvergencia $n \to \infty$ mellett értendő, a $\xrightarrow{\mathcal{D}}$ pedig az eloszlásbeli konvergenciát jelöli. A szemistabilis eloszlások a korlátlanul osztható eloszlásoknak egy, a stabilis eloszlásoknál (amikor is $k_n = n$-nek kell teljesülnie) bővebb részosztályát alkotják.

A disszertációban alapvetően a Csörgő, Haeusler, és Mason-től származó "probabilisztikus módszert" alkalmazzuk, melynek során elsőként a 2.1. tételben a szemistabilis eloszlások egy "probabilisztikus" leírását adjuk. A 2.3. tételben karakterizáljuk a vonzódó eloszlások kvantilisfüggvényeit, illetőleg a 2.3. korolláriumban azok eloszlásfüggvényeit. A kapott feltételek szükségesek és elégségesek. A 2.4. tételben konkrét példán keresztül bizonyítjuk, hogy a szemistabilis eloszlások geometriai parciális vonzástartománya szűkebb, mint a parciális vonzástartomány (amelyet akkor kapunk, ha a $\{k_n\}$ részsorozatra semmilyen feltételt nem teszünk). Foglalkozunk a fenti növekedési feltétel általánosabb formáival is.

A 2.5. tételben meghatározzuk azokat a részsorozatokat, amelyek mentén közös F eloszlásfüggvényű változók összegei konvergálnak, ha G egy szemistabilis, de nem stabilis eloszlás, és $F \in \mathbb{D}_{\mathrm{gp}}(G)$. Az eredményből következik egy $F \in \mathbb{D}_{\mathrm{gp}}(G)$ sztochasztikus kompaktsága. A 2.6. tétel az ún. összetartási tétel. Az összetartás jelensége alapvető fontosságú fogalom szemistabilis eloszlásokkal kapcsolatban: ha $F \in \mathbb{D}_{\mathrm{gp}}(G)$ egy G szemistabilis eloszlásra és F_n jelöli az $S_n(0,0) := \sum_{j=1}^{n} X_j$ összeg alkalmas centralizálás és normalizálás utáni eloszlásfüggvényét, akkor az F_n sorozat általában nem konvergens, csupán F_{k_n} az. Mégis, az F_n sorozat tagjai "összetartanak" alkalmasan választott, a G által származtatott, szintén szemistabilis eloszlásfüggvényekkel. Ez, és a fenti eredmények közül számos, teljes összegek helyett az $S_{k_n}(l,m) := \sum_{j=l+1}^{k_n-m} X_{j,k_n}$ *enyhén megnyírt* összegek esetére is igaz, ahol l és m két tetszés szerinti rögzített természetes szám. A választott "probabilisztikus módszer" fő előnye éppen az, hogy mind a teljes, mind az enyhén megnyírt összegek egységes keretben kezelhetőek segítségével.

A harmadik fejezetben az $S_n(l_n, m_n) = \sum_{j=l_n+1}^{n-m_n} X_{j,n}$ *közepesen megnyírt* összegek konvergenciájával foglalkozunk, ahol l_n és m_n olyan számsorozatok, hogy $l_n \to \infty$, $\frac{l_n}{n} \to 0$, és $m_n \to \infty$, $\frac{m_n}{n} \to 0$. Fő eredményként (3.1. tétel) igazoljuk, hogy ha $F \in \mathbb{D}_{\mathrm{gp}}(G)$ és G egy enyhe folytonossági feltételnek eleget tevő, nem aszimmetrikus szemistabilis eloszlás, akkor $S_n(l_n, m_n)$ aszimptotikusan normális az l_n és m_n számsorozatok tetszőleges választása esetén. Ha az enyhe folytonossági feltétel nem teljesül, az aszimptotikus normalitás akkor is elérhető l_n és m_n alkalmas választásával (3.2. tétel). Az értekezés 3.3. tételében megmutatjuk, hogy a teljes összeg aszimptotikus szemistabilitása lényegében véve tetszőlegesen csekély részarányú fölső és alsó rendstatisztikákon múlik, mivel a minta középső része a teljes összeghez szükséges mértékű normalizációnál eltűnik.

A 4. fejezetben max-szemistabilis eloszlásokkal és azok geometriai parciális max-vonzástartományával foglalkozunk. Bevezetve az $M_n := X_{n,n} = \max\{X_1, \ldots, X_n\}$ jelölést, akkor mondjuk, hogy a G eloszlás max-szemistabilis, az F pedig a G geometriai parciális max-vonzástartományába tartozik, jelben $F \in \mathbb{MD}_{\mathrm{gp}}(G)$, ha alkalmas a_n és $b_n > 0$ centralizáló és normalizáló konstansokkal, egy, az előző növekedési feltételt kielégítő $\{k_n\}$ részsorozat mentén

$$\mathbb{P}(b_n^{-1}(M_{k_n} - a_n) \leq x) = \Big(F\big(b_n x + a_n\big)\Big)^{k_n} \to G(x), \qquad x \in C_G.$$

Itt C_G a G eloszlásfüggvény folytonossági pontjait jelöli. Ismert, hogy egy eltolás- és skálatranszformáció erejéig a max-szemistabilis eloszlások az alábbi három osztály valamelyikéhez tartoznak:

$$\begin{aligned}
\Lambda_\nu(x) &= \exp\big(-\exp(-x)\nu(x)\big), \quad x \in \mathbb{R},\\
\Phi_{\alpha,\nu}(x) &= \exp\big(-x^{-\alpha}\nu(\log x)\big), \qquad x \in (0,\infty),\\
\Psi_{\alpha,\nu}(x) &= \exp\Big(-|x|^\alpha \nu\big(\log|x|\big)\Big), \qquad x \in (-\infty, 0),
\end{aligned}$$

ahol $\nu(\cdot)$ egy pozitív periodikus függvény. A 4.1–4.3. tételekben matematikailag teljes karakterizációt adunk a max-szemistabilis eloszlások geometriai parciális max-vonzástartományaira, egyúttal egy ellenpéldával megmutatjuk az irodalomban korábban fellelhető karakterizáció hibás voltát.

A 4.4. tétel a maximumokra vonatkozó összetartási tétel: amíg az összegek esetén az összetartás az egész számegyenesen egyenletesen is teljesült, itt csak Lévy-távolságban történő összetartást lehet állítani, de a legtöbb esetben ez is elegendő. A 4.6. tételben pedig meghatározzuk azokat a részsorozatokat, melyek mentén konvergencia történik: a kapott feltételek ismét szükségesek és elegendőek.

Az 5. fejezet majdnem biztos határeloszlástételekkel foglalkozik, a korábbi összetartási tételeknek mutatjuk egy alkalmazását. Legyen $A_n > 0$ és B_n két számsorozat, $S_n = X_1 + \cdots + X_n$, és jelölje $I(\cdot)$ az indikátorfüggvényt. Sokan vizsgálták a

$$\mathbb{P}\left\{\frac{S_n - B_n}{A_n} \le x\right\} \to G(x), \qquad \text{minden } x \in C_G \text{ esetén,}$$

eloszlásbeli, "gyenge" határeloszlástétel és a

$$\frac{1}{\log n}\sum_{r=1}^{n}\frac{1}{r}\, I\left\{\frac{S_r - B_r}{A_r} \le x\right\} \to G(x), \qquad \text{m.b. minden } x \in C_G \text{ esetén,}$$

majdnem biztos, "erős" határeloszlástétel közötti kapcsolatot. Kiderült, hogy enyhe feltételek mellett a "gyenge" tételből következik az "erős" állítás: a disszertációban megmutatjuk, hogy eloszlások egy széles osztályára, nevezetesen a szemistabilis eloszlások geometriai parciális vonzástartományába tartozó eloszlásokra annak ellenére igaz a *teljes* $\{n\}$ *mentén* egy "erős" határeloszlástétel, hogy a megfelelő "gyenge" nem teljesül. Nem is teljesülhet, hiszen a szemistabilitás lényege, hogy eloszlásbeli konvergencia csak $\{k_n\}$, és nem pedig az $\{n\}$ mentén van. Ez az 5.1. tétel tartalma. Az 5.2. tételben meghatározzuk a lehetséges majdnem biztos limeszek $\mathcal{J}^*$ halmazát, ha az 5.1. tételbeli, az idáig felállított elméletből természetesen adódó normálókonstansokat egy tetszőleges normálósorozat váltja fel. Az 5.3. tételben pedig megmutatjuk, hogy az így kapott osztályból a részsorozatonkénti limeszképzés sem vezet ki, sőt, létezik egy olyan "univerzális" normálósorozat, hogy az adódó parciális limeszek halmaza a teljes $\mathcal{J}^*$.

Az 5.4. és az 5.5. tételben analóg eredményeket bizonyítunk összegek helyett maximumokra. Noha a szemistabilis és a max-szemistabilis eloszlásokra vonatkozó állítások feltűnő matematikai szimmetriát mutatnak, az alkalmazott módszerek mind a korábbiak során, mind most gyökeresen eltérőek. Az 5.4. tételben már azonnal az általános normálósorozatra vonatkozó eredményt igazoljuk, míg az 5.5. tétel a lehetséges limeszként adódó $\mathcal{K}^*$ osztály részsorozatonkénti limeszképzésre vonatkozó zártságát állítja. Igazoljuk, hogy $\mathcal{K}^*$ is előáll egy alkalmas "univerzális" normálósorozat segítségével kapott parciális limeszek halmazaként.